20.330J Fields, Forces and Flows in Biological Systems

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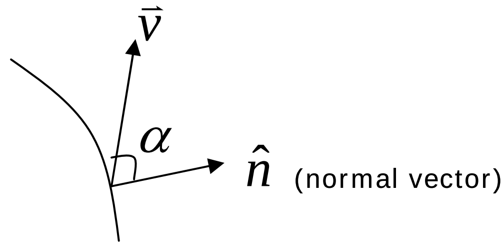
Review: Vector Calculus

Vector Product

$$\vec{v}\cdot\hat{n} = v_x n_x + v_y n_y + v_z n_z$$
$$= |\vec{v}||\hat{n}|\cos(\alpha)$$



Gradient (on a scalar function)

$$\vec{\nabla} = \hat{i}_x \frac{\partial}{\partial x} + \hat{i}_y \frac{\partial}{\partial y} + \hat{i}_z \frac{\partial}{\partial z}$$

$$\vec{\nabla} p = \hat{i}_x \frac{\partial p}{\partial x} + \hat{i}_y \frac{\partial p}{\partial y} + \hat{i}_z \frac{\partial p}{\partial z}$$

Divergence (operated on vector)

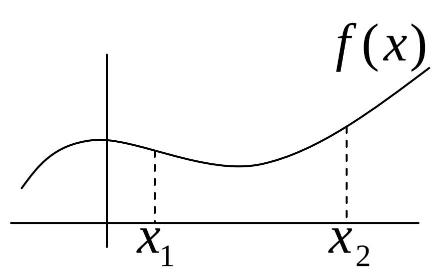
$$\vec{\nabla}\cdot\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$ => scalar

Curl (operated on vector)

$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$ => vector

In 1D integration…

$$f(x_2) - f(x_1) = \int_{x_1}^{x_2} \frac{\partial f}{\partial x} dx$$



…similarly, we have two different integral theorems for vector calculus.

(1) Gauss' theorem (Divergence theorem)

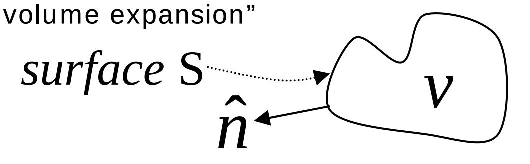
For any vector field $\hat{v}$,

$$\oint_s \vec{v} \cdot \hat{n}\, da = \int_v (\nabla \cdot \vec{v}) dv$$

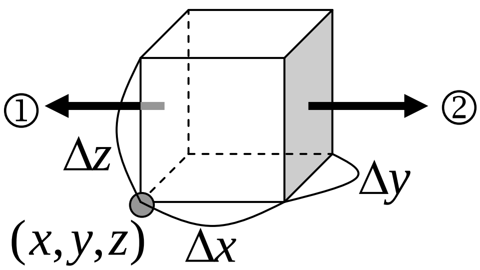
$velocity \times area$

"total outgoing volume flow rate"

"volume expansion"



Proof: consider infinitesimal cube.



From surfaces ① and ②:

$$\oint_s (\vec{v} \cdot \hat{n})\, da \rightarrow (V_x\big|_{x+\Delta x} - V_x\big|_x)\Delta y \Delta z$$

Similarly, from other surfaces,

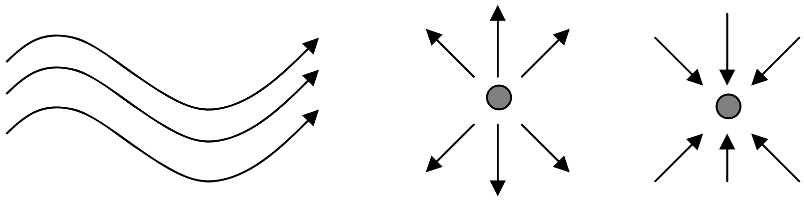
$$\begin{aligned}\oint_s (\vec{v} \cdot \hat{n})\, da = & (V_x\big|_{x+\Delta x} - V_x\big|_x)\Delta y \Delta z \\ & + (V_y\big|_{y+\Delta y} - V_y\big|_y)\Delta x \Delta z \\ & + (V_z\big|_{z+\Delta z} - V_z\big|_z)\Delta x \Delta y\end{aligned}$$

Divide each terms with Δx, Δy, Δz respectively,

$$\begin{aligned}&= \left[\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right]\Delta x \Delta y \Delta z \\ &= \oint_V (\nabla \cdot \vec{V}) dV\end{aligned}$$

Meaning of "$\nabla \cdot \vec{V}$"

- volume expansion
- net outgoing flux
- for incompressible flow, $\nabla \cdot \vec{V} = 0$ (no fluid source/sink)

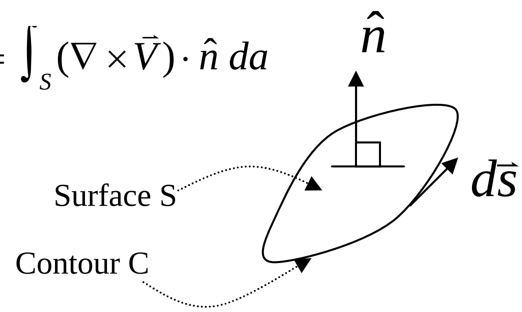


$\nabla \cdot \vec{V} = 0$ "divergence" free $\qquad \nabla \cdot \vec{V} > 0 \qquad \nabla \cdot \vec{V} < 0$

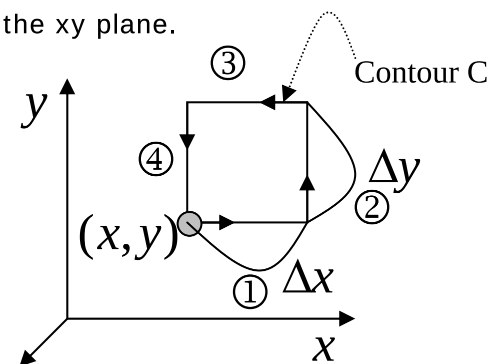
(2) Stokes' theorem (curl theorem)

For a given vector field $\hat{v}$,

$$\oint_C \vec{V} \cdot d\vec{s} = \int_S (\nabla \times \vec{V}) \cdot \hat{n}\, da$$



Proof: think about the rectangle in the xy plane.



$$\oint_C \vec{V} \cdot d\vec{s}$$

① ③

$$= (V_x\big|_y - V_x\big|_{y+\Delta y})\Delta x$$

$$+(V_y\big|_{x+\Delta x} - V_y\big|_x)\Delta y$$

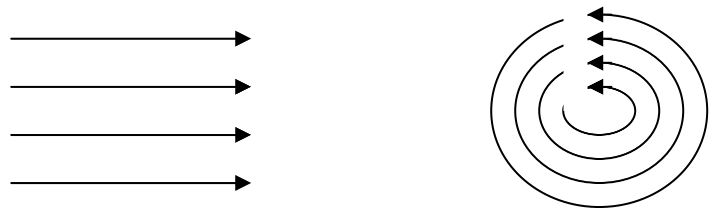
$$= \left(-\frac{V_x\big|_{y+\Delta y} - V_x\big|_y}{\Delta y} + \frac{V_y\big|_{x+\Delta x} - V_y\big|_x}{\Delta x} \right) \Delta x \Delta y$$

$$= \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \Delta x \Delta y = \left(\nabla \times \vec{V} \right)_z \Delta x \Delta y$$

Similar for curves in other planes...

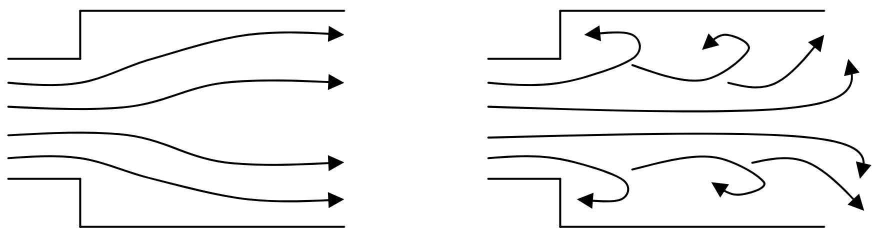
Meaning of "$\nabla \times \vec{V}$"

- Represents "circulation" of the flow.



$\nabla \times \vec{V} = 0$ Laminar flow

$\nabla \times \vec{V} \neq 0$ Turbulent flow



References

- H&M website: Chapter 2
- Appendix of TY & K