Discretization of the Poisson Problem in $\mathbb{R}^1$: Theory and Implementation

April 7 & 9, 2003

Theory

Goals

A priori...

A priori error estimates: N1

bound various "measures"

of u [exact] $-$ u_h [approximate];

in terms of $C(\Omega,$ problem parameters$)$,

h [mesh diameter], and u.

Theory

Goals

...A priori...

u: $-u_{xx} = f,\ u(0) = u(1) = 0$

$$a(u,v) = \ell(v), \qquad \forall\, v \in X$$

$$a(w,v) = \int_0^1 w_x\, v_x\, dx, \qquad \ell(v) = \text{``}\int_0^1 f\, v\, dx\text{''}$$

$$X = \{v \in H^1(\Omega) \,|\, v(0) = v(1) = 0\}$$

Theory

Goals

...A priori

u_h:

$$a(u_h, v) = \ell(v), \qquad \forall\, v \in X_h$$

$$a(w,v) = \int_0^1 w_x\, v_x\, dx, \qquad \ell(v) = \text{“}\int_0^1 f\, v\, dx\text{”}$$

$$X_h = \{v \in X \mid v|_{T_h} \in \mathbb{P}_1(T_h), \quad \forall\, T_h \in \mathcal{T}_h\}$$

Theory

Goals

A posteriori

A posteriori error estimates: **N2**

bound various "measures"

of u [exact] $-$ u_h [approximate];

in terms of $C(\Omega,$ problem parameters$)$,

h [mesh diameter], and u_h.

Theory

Projection

Definition

Given Hilbert spaces $\boldsymbol{Y}$ and $\boldsymbol{Z} \subset \boldsymbol{Y}$,

$$(\underbrace{\Pi y}_{\in Z}, v)_Y = (\underbrace{y}_{\in Y}, v)_Y, \qquad \forall\, v \in Z$$

defines the *projection* of y onto Z, Πy;

$$\Pi\colon\ Y \to Z\ .$$

Theory

Projection

Property

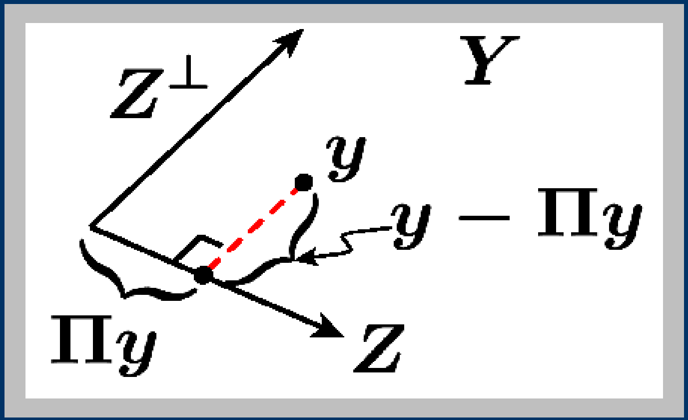
The projection Πy minimizes $\|y - z\|_Y^2, \quad \forall\, z \in Z$.

Why?

$$\|y - (\underbrace{\Pi y + v}_{\text{any } z \in Z})\|_Y^2 = ((y - \Pi y) - v, (y - \Pi y) - v)_Y$$

$$= \|y - \Pi y\|_Y^2 - 2\underbrace{(y - \Pi y, v)_Y}_{0:\ v \in Z} + \|v\|_Y^2, \quad \forall\, v \in Z\ .$$

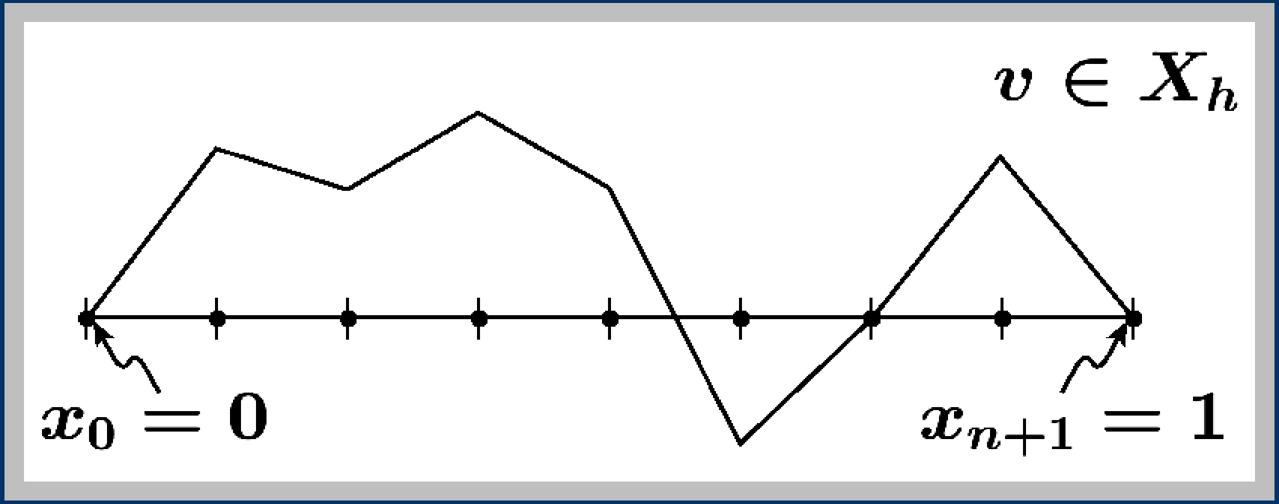
Geometry of projection:



Orthogonality: $(y - \Pi y, v)_Y = 0, \qquad \forall\, v \in Z.$ **E1**

Recall

$$X_h = \{ v \in X \mid v|_{T_h} \in \mathbb{P}_1(T_h), \quad \forall\, T_h \in \mathcal{T}_h \}$$



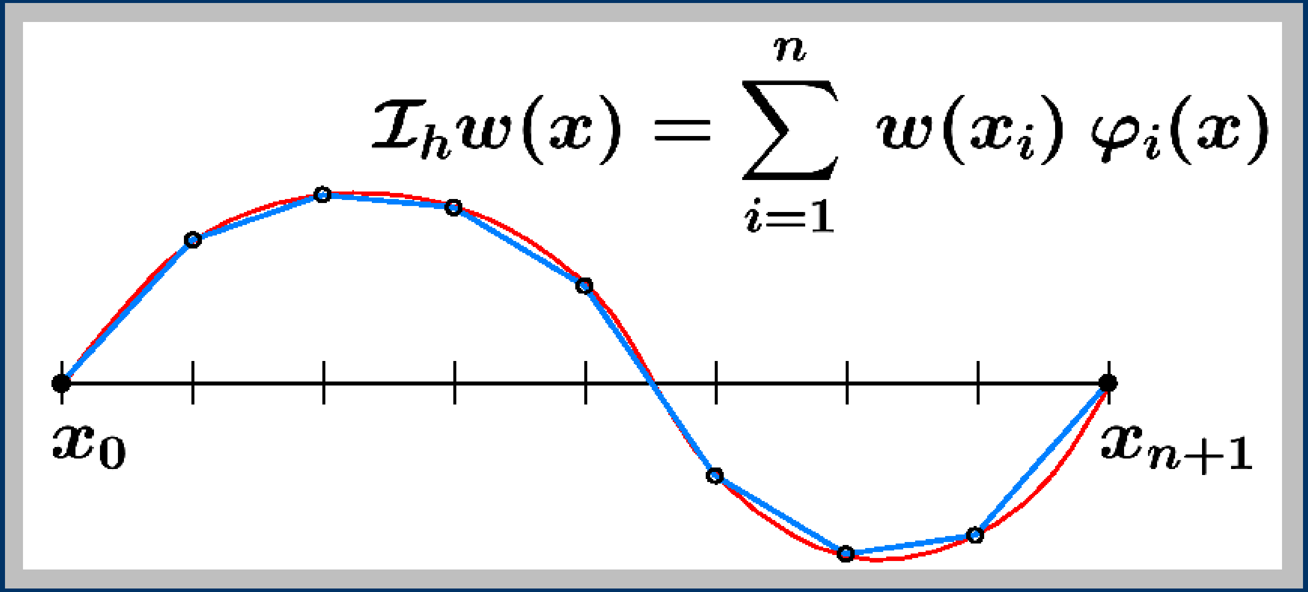
Theory

The Interpolant

...Definition

Given $w \in X$, the *interpolant* $\mathcal{I}_h w$ satisfies:

$$\mathcal{I}_h w \in X_h; \text{ and } \mathcal{I}_h w(x_i) = w(x_i), \quad i = 0, \ldots, n+1\,.$$



Theory

The Interpolant

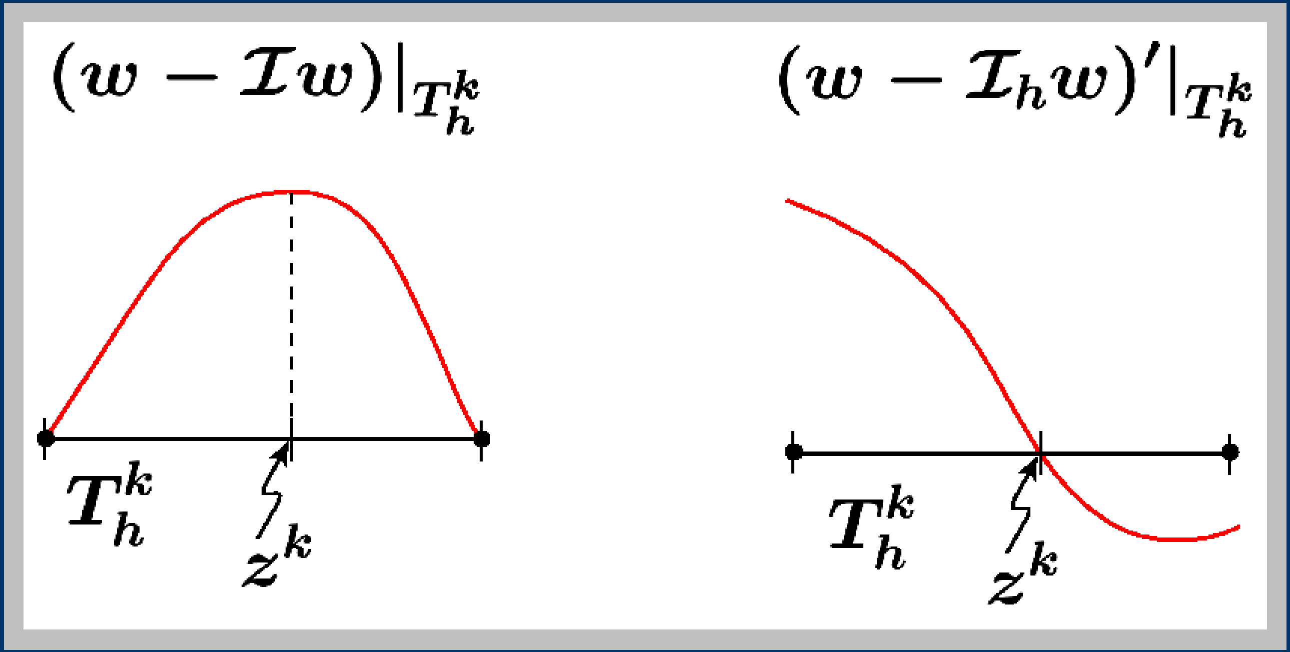
Approximation Theory...

If $w \in X$, and $w|_{T_h} \in C^2(T_h)$, $\forall\, T_h \in \mathcal{T}_h$, then

$$|w - \mathcal{I}_h w|_{H^1(\Omega)} \leq h \max_{T_h \in \mathcal{T}_h} \left(\max_{x \in T_h} |w''| \right)$$

$$\|w - \mathcal{I}_h w\|_{L^2(\Omega)} \leq h^2 \max_{T_h \in \mathcal{T}_h} \left(\max_{x \in T_h} |w''| \right) .$$

Sketch of proof:



Theory

The Interpolant

...Approximation Theory...

$$\left| (w - \mathcal{I}_h w)'|_{T_h^k}(x) \right| = \left| \int_{z^k}^{x} (w - \mathcal{I}_h w)''|_{T_h^k}\, dx \right| = \left| \int_{z^k}^{x} w''\, dx \right|$$

$$\leq h \max_{x \in T_h^k} |w''|$$

$$\sum_{k=1}^{K} \int_{T_h^k} (w - \mathcal{I}_h w)'|_{T_h^k}^2\, dx \leq \frac{1}{h} h \left(h \max_{k=1,\ldots,K} \max_{x \in T_h^k} |w''| \right)^2$$

E2

Theory

The Interpolant

...Approximation Theory

If $w \in X$, and $w \in H^2(\Omega, \mathcal{T}_h)$,

$$|w - \mathcal{I}_h w|_{H^1(\Omega)} \leq \frac{h}{\pi} \|w\|_{H^2(\Omega,\mathcal{T}_h)}$$

$$\|w - \mathcal{I}_h w\|_{L^2(\Omega)} \leq \frac{h^2}{\pi^2} \|w\|_{H^2(\Omega,\mathcal{T}_h)} \;,$$

where

$$\|w\|^2_{H^2(\Omega,\mathcal{T}_h)} \equiv \sum_{k=1}^{K} \|w\|^2_{H^2(T_h^k)} = \sum_{k=1}^{K} \int_{T_h^k} w_{xx}^2 + w_x^2 + w^2 \, dx \;.$$

Theory

Error: Energy Norm

Definition...

Define the energy, or "a", norm $|||v|||$ as

$$|||v|||^2 = a(v, v) \qquad \text{(generally)}$$

$$= \int_0^1 v_x^2 \, dx \; = \; |v|^2_{H^1(\Omega)} \qquad \text{(here)} \quad .$$

Note: $|||\cdot|||$ is *problem-dependent*.

Theory

Error: Energy Norm

...Definition

Of interest: for

$u(x)$ (exact solution)

$u_h(x)$ (finite element approximation)

$\Rightarrow e(x) = (u - u_h)(x)$ (discretization error)

find bound for $|||e|||$ in terms of h, u.

Theory

Error: Energy Norm

Orthogonality

Since $a(u,v) = \ell(v), \ \forall\, v \in X$

then $a(u,v) = \ell(v), \ \forall\, v \in X_h$ $\quad (X_h \subset X)$,

but $-\ [a(u_h,v) = \ell(v)], \ \forall\, v \in X_h$

so $a(u - u_h, v) = 0, \ \forall\, v \in X_h$ (bilinearity).

Theory

Error: Energy Norm

General Bound...

For any $w_h = u_h + v_h \in X_h$, $\qquad v_h \in X_h$

$$\underbrace{a(u - w_h, u - w_h)}_{|||u-w_h|||^2} = a\left((u - u_h) - v_h, (u - u_h) - v_h\right)$$

$$= \underbrace{a(u - u_h, u - u_h)}_{|||e|||^2} \underbrace{- 2a(u - u_h, v_h)}_{0\text{: orthogonality}} + \underbrace{a(v_h, v_h)}_{>0 \text{ if } v_h \neq 0}$$

$\Rightarrow$

$$|||e||| = \inf_{w_h \in X_h} |||u - w_h||| \, .$$

Theory

Error: Energy Norm

...General Bound...

In words: even if you *knew* u,

you could not find a w_h in X_h

more accurate than u_h

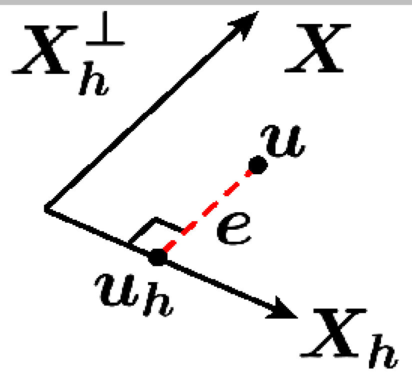
in the energy norm.

Theory

Error: Energy Norm

...General Bound...

Geometry



$$a(u - u_h, v) = 0, \quad \forall v \in X_h$$

$\Rightarrow u_h = \Pi_h^a \, u$: the projection of (closest point to) u on X_h in the a norm.

Theory

Error: Energy Norm

...General Bound

Miracle ?: $a(\underbrace{\Pi_h^a u}_{u_h}, v) = a(u, v), \; \forall \, v \in X_h$;

but we do not know u . . .

NO:

$$a(u, v) = \underbrace{\ell(v)}_{\textit{can}\ \text{evaluate}} \Rightarrow a(\underbrace{\Pi_h^a u}_{u_h}, v) = \ell(v), \; \forall \, v \in X_h \, .$$

Only in the energy inner product can we compute $\Pi_h \, u$ without knowing u. N3

Theory

Error: Energy Norm

Particular Bound

We know $\quad |u - \mathcal{I}_h u|_{H^1(\Omega)} \leq \dfrac{h}{\pi} \|u\|_{H^2(\Omega,\mathcal{T}_h)}$.

Thus

$$|||e||| = \inf_{w_h \in X_h} |||u - w_h||| \leq |||u - \mathcal{I}_h u|||$$
$$= |u - \mathcal{I}_h u|_{H^1(\Omega)} \leq \frac{h}{\pi} \|u\|_{H^2(\Omega,\mathcal{T}_h)}$$

E3 N4

(assuming $\|u\|_{H^2(\Omega,\mathcal{T}_h)}$ finite).

Theory

Error: H^1 Norm

Reminders...

The H^1 *norm*:

$$\|v\|^2_{H^1(\Omega)} = |v|^2_{H^1(\Omega)} + \|v\|^2_{L^2(\Omega)}$$

$$= \int_0^1 v_x^2 \, dx + \int_0^1 v^2 \, dx \; ;$$

$\|e\|_{H^1(\Omega)}$ measures e and e_x.

Theory

Error: H^1 Norm

...Reminders

Coercivity of $a(\cdot,\cdot)$:

$\exists \alpha > 0$ such that $$a(v,v) \geq \alpha \, \|v\|^2_{H^1(\Omega)}, \quad \forall\, v \in X$$

$$\left(\int_0^1 v_x^2 \, dx \geq \alpha \left(\int_0^1 v_x^2 \, dx + \int_0^1 v^2 \, dx \right) \right) .$$

Continuity of $a(\cdot,\cdot)$:

$\exists \beta \; (= 1) > 0$ such that $a(w,v) \leq \beta \|w\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$.

Theory

Error: H^1 Norm

General Result

The error $e = u - u_h$ satisfies

$$\|e\|_{H^1(\Omega)} \leq \underbrace{(1 + \frac{\beta}{\alpha})}_{\text{degradation}} \underbrace{\inf_{w \in X_h} \|u - w_h\|_{H^1(\Omega)}}_{\text{error in } H^1 \text{ projection of } u \text{ on } X_h} \;;$$

in general u_h is *not* the H^1 projection of u on X_h.

E4 N5

Theory

Error: H^1 Norm

Particular Result

We know $\|u - \mathcal{I}_h u\|_{H^1(\Omega)} \leq \sqrt{2}\,\dfrac{h}{\pi}\,\|u\|_{H^2(\Omega,\mathcal{T}_h)}$. Thus

$$\|e\|_{H^1(\Omega)} = \left(1 + \frac{\beta}{\alpha}\right) \inf_{w_h \in X_h} \|u - w_h\|_{H^1(\Omega)}$$

$$\leq \left(1 + \frac{\beta}{\alpha}\right) \|u - \mathcal{I}_h u\|_{H^1(\Omega)}$$

$$\leq \sqrt{2}\left(1 + \frac{\beta}{\alpha}\right) \frac{h}{\pi}\,\|u\|_{H^2(\Omega,\mathcal{T}_h)} \ .$$

Theory

Error: L^2 Norm

Reminder

The L^2 *norm*:

$$\|v\|_{L^2(\Omega)} = \left(\int_0^1 v^2 \, dx \right)^{1/2} ;$$

$\|e\|_{L^2(\Omega)}$ measures e.

Theory

Error: L^2 Norm

Particular Result

The L^2 error satisfies

$$\|e\|_{L^2(\Omega)} \leq C\, h\, \|e\|_{H^1(\Omega)}$$

$$\leq C\, h^2\, \|u\|_{H^2(\Omega,\mathcal{T}_h)}\ ,$$

for C independent of h and u.

N6

Theory

A linear-functional "output" s is defined by

$$s = \ell^O(u) + c^O \; ;$$

where

$$\ell^O \colon \;\; H_0^1(\Omega) \to \mathbb{R}$$

is a bounded linear functional

$$|\ell^O(v)| \leq C \, \|v\|_{H^1(\Omega)} \; , \qquad \forall \, v \in H_0^1(\Omega) \; .$$

Theory

Linear Functionals

...Motivation...

Very relevant: engineering quantities of interest.

For example:

s: average over $\mathcal{D} \subset \Omega$, with

$$\ell^O(v) = \int_{\mathcal{D}} v \, dx \; ;$$

s: flux at boundary, $u_x(0)$, with

$$\ell^O(v) = -\int_0^1 (1-x)_x \, v_x, \; c^O = \int_0^1 f(1-x) \, dx \; .$$

N7

Theory

Linear Functionals

...Motivation

Of interest: $s = \ell^O(u) + c^O$,

$$s_h = \underbrace{\ell^O\,(u_h) + c^O}_{\text{finite element prediction of output}};$$

error in output is thus

$$|s - s_h| = |\ell^O(u) - \ell^O(u_h)| = |\ell^O(u - u_h)|$$

$$= |\ell^O(e)|\,.$$

Theory

Linear Functionals

General Result...

If $\ell^O \in H^{-1}(\Omega)$, then

$$|\ell^O(e)| \leq C \, \|e\|_{H^1(\Omega)} \text{ (boundedness).}$$

If $\ell^O \in L^2(\Omega)$, then

$$|\ell^O(e)| \leq C \, \|e\|_{L^2(\Omega)} \text{ (boundedness).}$$

In fact: for any $\ell^O \in H^{-1}(\Omega)$,

$$|\ell^O(e)| \leq C \, \|e\|_{H^1(\Omega)} \, \|\psi - \psi_h\|_{H^1(\Omega)}$$

where

$$a(v, \psi) = -\ell^O(v), \qquad \forall \, v \in X$$

N8

$$a(v, \psi_h) = -\ell^O(v), \qquad \forall \, v \in X_h \, ,$$

and ψ is an adjoint, or dual, variable.

Theory

Linear Functionals

Particular Result

From our earlier bounds for $\|e\|_{H^1(\Omega)}$ and $\|e\|_{L^2(\Omega)}$ for linear finite elements:

for $\quad \ell^O \in H^{-1}(\Omega)$: $\quad |\ell^O(e)| \leq C\, h\, \|u\|_{H^2(\Omega,\mathcal{T}_h)}$

for $\quad \ell^O \in L^2(\Omega)$: $\quad |\ell^O(e)| \leq C\, h^2\, \|u\|_{H^2(\Omega,\mathcal{T}_h)}$.

Better yet: for $\ell^O \in H^{-1}(\Omega)$

$$|\ell^O(e)| \leq C\, \boxed{h^2}\, \|u\|_{H^2(\Omega,\mathcal{T}_h)}\, \|\psi\|_{H^2(\Omega,\mathcal{T}_h)} .$$

Implementation

Four steps:

A Proto-Problem,

Elemental Quantities;

Assembly;

Boundary Conditions;

and Quadrature.

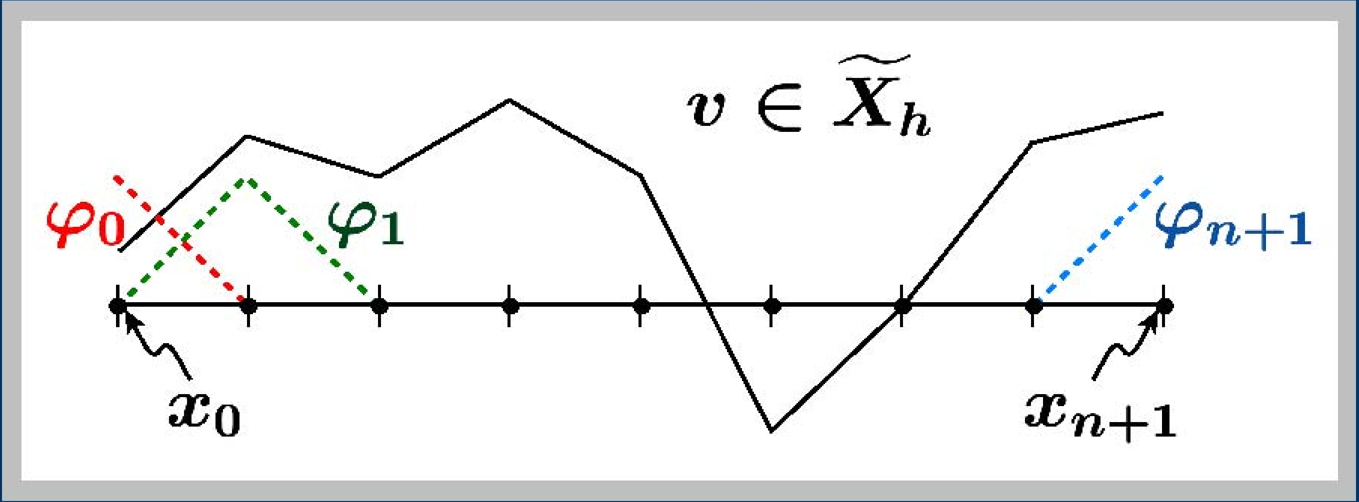
Implementation

A Proto-Problem

Space and Basis

Let $\widetilde{X}_h = \{v \in H^1(\Omega) \mid v|_{T_h} \in \mathbb{P}_1(T_h),\ \forall\, T_h \in \mathcal{T}_h\}$

$$= \text{span}\,\{\varphi_0, \ldots, \varphi_{n+1}\}\,.$$



Implementation

A Proto-Problem

Definition

"Find" $\tilde{u}_h \in \widetilde{X}_h$ such that

$$a(\tilde{u}_h, v) = \ell(v), \qquad \forall\, v \in \widetilde{X}_h \,.$$

We never actually solve this problem:

it serves only as a convenient pre-processing step.

Implementation

A Proto-Problem

Discrete Equations...

$$\underline{\widetilde{A}}_h \, \underline{\tilde{u}}_h = \underline{\widetilde{F}}_h \qquad\qquad \tilde{u}_h(x) = \sum_{i=0}^{n+1} \tilde{u}_{h\,i}\, \varphi_i(x)$$

$$\widetilde{A}_{h\,i\,j} = a(\varphi_i, \varphi_j) = \int_0^1 \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx}\, dx, \quad 0 \le i, j \le n+1$$

$$\widetilde{F}_{h\,i} = \ell(\varphi_i) \left(= \int_0^1 f\, \varphi_i \, dx \right), \qquad 0 \le i \le n+1$$

Implementation

A Proto-Problem

...Discrete Equations

Matrix form:

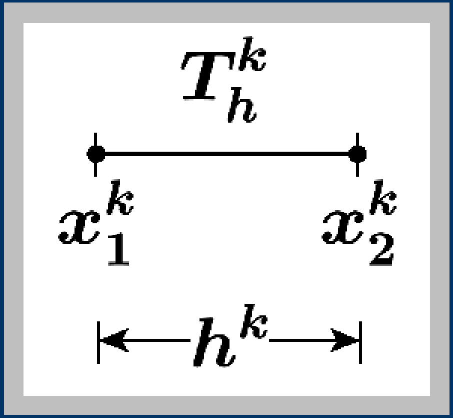
$$\widetilde{A}_h = \frac{1}{h}\begin{pmatrix} 1 & -1 & & & & & \\ -1 & 2 & -1 & & & 0 & \\ & -1 & 2 & -1 & & & \\ & & & \ddots & & & \\ & 0 & & & -1 & 2 & -1 \\ & & & & & -1 & 1 \end{pmatrix}$$

Implementation

Elemental Quantities

Local Definitions

Element T_h^k: $\mapsto x$



x_1^k: local node 1 of element T_h^k;

x_2^k: local node 2 of element T_h^k;

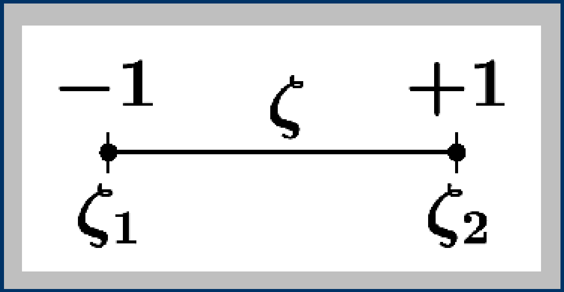
h^k: length of element T_h^k.

Implementation

Elemental Quantities

Reference Element...

Definition: $\widehat{T} = (-1, 1)$



ζ_1: reference element node 1;

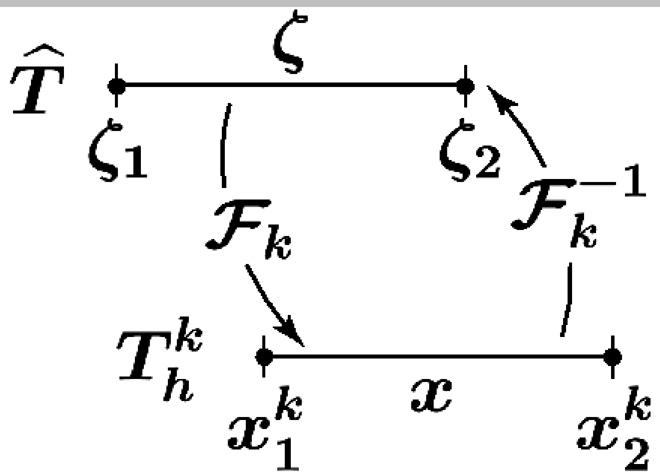
ζ_2: reference element node 2.

Implementation

Elemental Quantities

...Reference Element

Relation of $\widehat{T}$ *to each* T_h^k: Affine Mappings



$$\mathcal{F}_k(\zeta) = x_1^k + \tfrac{1}{2}\,(1+\zeta)\,h^k$$

$$\mathcal{F}_k^{-1}(x) = 2\;\frac{x - x_1^k}{h^k} \;-\; 1$$

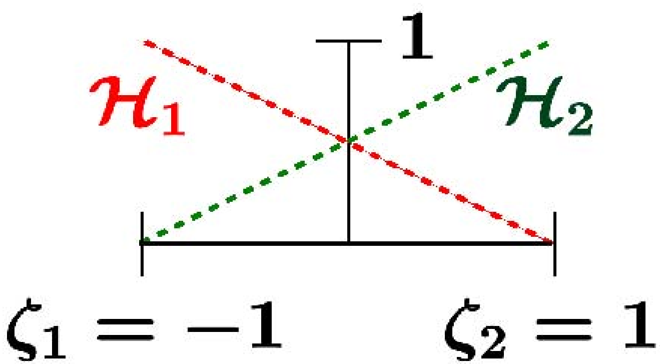
Implementation

Elemental Quantities

Reference Element Space, Basis

Define *space* $\widehat{X} = \mathbb{P}_1(\widehat{T})$: all linear polynomials over $\widehat{T}$; $\dim(\widehat{X}) = 2$.

Introduce *basis* for $\widehat{X}$, $\mathcal{H}_1(\zeta), \mathcal{H}_2(\zeta)$:



$$\left.\begin{aligned} \mathcal{H}_1(\zeta) &= \frac{(1-\zeta)}{2} \\ \mathcal{H}_2(\zeta) &= \frac{(1+\zeta)}{2} \end{aligned}\right\} \text{ Lagrangian interpolants}$$

Implementation

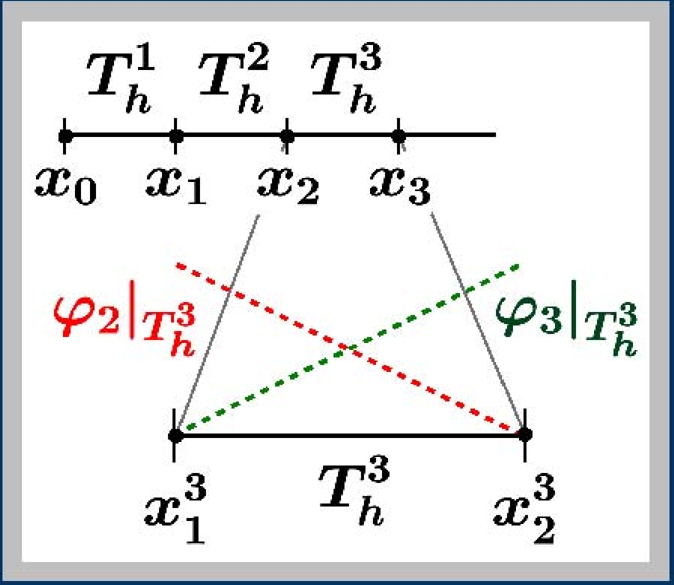
Elemental Quantities

Elemental Matrices...

$$\widetilde{A}_{h\,i\,j} = a(\varphi_i, \varphi_j) = \int_0^1 \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}\,dx$$

Element T_h^3 (say) contributes

$$\int_{T_h^3} \left.\frac{d\varphi_{2 \text{ or } 3}}{dx}\right|_{T_h^3} \left.\frac{d\varphi_{2 \text{ or } 3}}{dx}\right|_{T_h^3} dx$$



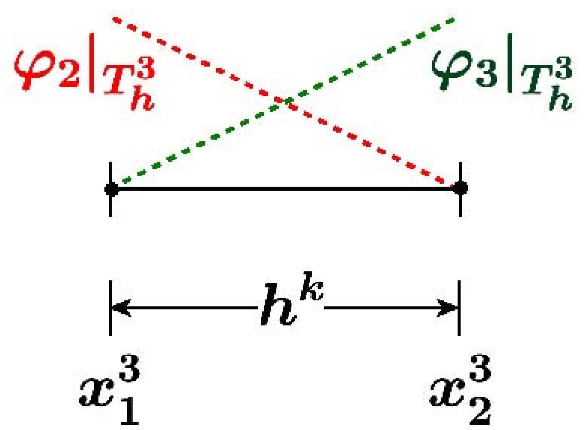
Implementation

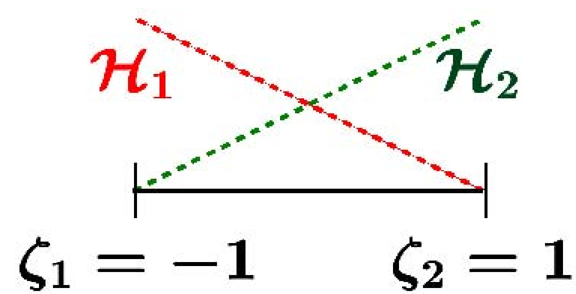
Elemental Quantities

...Elemental Matrices...

Change variables $T_h^3 \rightarrow \widehat{T}$:

N9





$$\int_{T_h^3} \frac{d\varphi_{2 \text{ or } 3}}{dx}\,\frac{d\varphi_{2 \text{ or } 3}}{dx}\,dx \qquad \int_{-1}^{1}\left(\frac{d\mathcal{H}_{1 \text{ or } 2}}{d\zeta}\,\frac{2}{h^k}\right)\left(\frac{d\mathcal{H}_{1 \text{ or } 2}}{d\zeta}\,\frac{2}{h^k}\right)\left(d\zeta\frac{h^k}{2}\right)$$

Implementation

Elemental Quantities

...Elemental Matrices

Define $\underline{A}^k \in \mathbb{R}^{2\times 2}$ (e.g., $k = 3$):

E5 E6 N10

$$\frac{2}{h^k} \int_{-1}^{1} \frac{d\mathcal{H}_{\alpha(1 \text{ or } 2)}}{d\zeta} \frac{d\mathcal{H}_{\beta(1 \text{ or } 2)}}{d\zeta}\, d\zeta =$$

$$\frac{2}{h^k} \int_{-1}^{1} \frac{d}{d\zeta}\underline{\mathcal{H}}\, \frac{d}{d\zeta}\underline{\mathcal{H}}^T\, d\zeta =$$

$$\underline{\mathcal{H}} = \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{pmatrix}$$

$$\frac{1}{h^k}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \equiv \underline{A}^k$$

Elemental Stiffness Matrix

Implementation

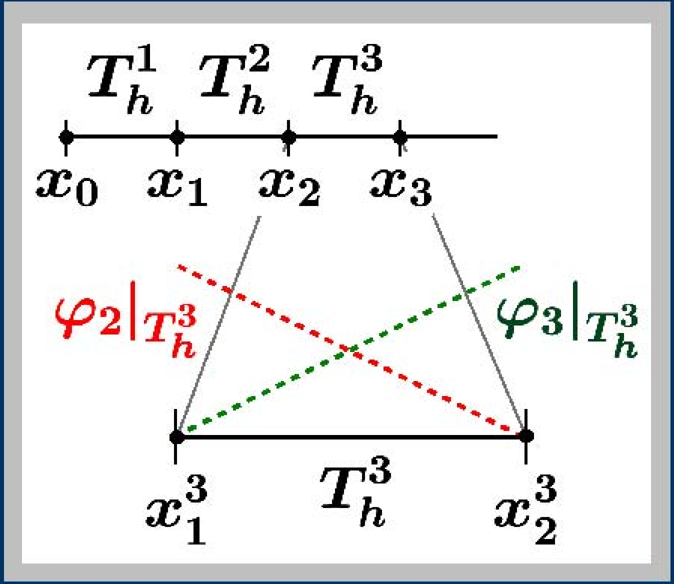
Elemental Quantities

Elemental "Loads"...

$$\widetilde{F}_{h\,i} = \ell(\varphi_i) \underset{\text{(say)}}{=} \int_0^1 f\,\varphi_i\,dx$$

Element T_h^3 (say) contributes

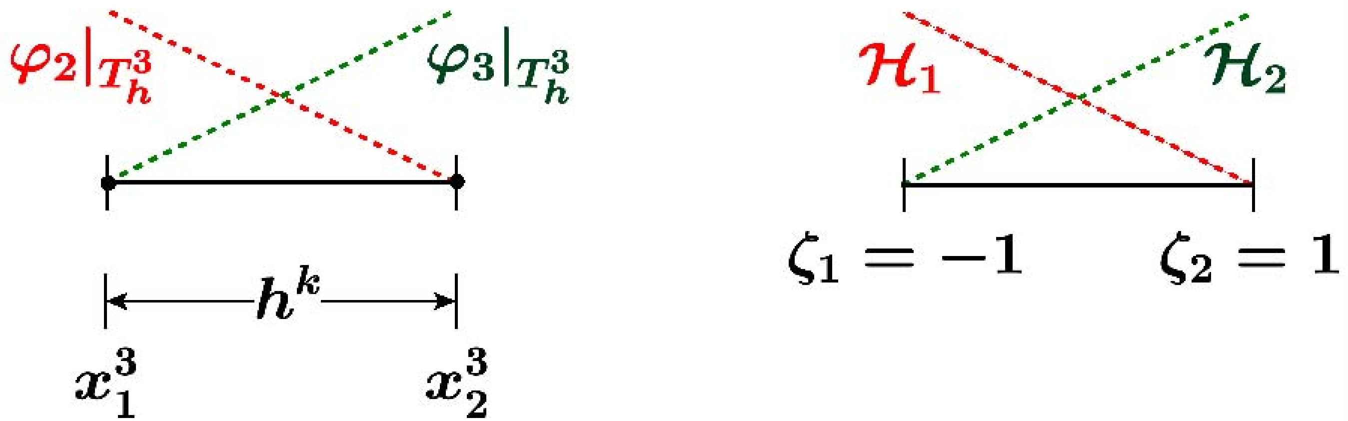
$$\int_{T_h^3} f\,\varphi_{2 \text{ or } 3}\,dx$$



Elemental Quantities

...Elemental "Loads"...

Change variables $T_h^3 \rightarrow \widehat{T}$:



$$\int_{T_h^3} f \, \varphi_{2 \text{ or } 3} \, dx \qquad\qquad \frac{h^k}{2} \int_{-1}^{1} f \, \mathcal{H}_{1 \text{ or } 2} \, d\zeta$$

Implementation

Elemental Quantities

...Elemental "Loads"

Define $\underline{F}^k \in \mathbb{R}^2$ (e.g., $k = 3$):

$$F^k_\alpha = \frac{h^k}{2} \int_{-1}^{1} f \, \mathcal{H}_{\alpha(1 \text{ or } 2)} \, d\zeta$$

Elemental Load Vector

$$= \frac{h^k}{2} \int_{-1}^{1} f \, \underline{\mathcal{H}} \, d\zeta \qquad \underline{\mathcal{H}} = \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{pmatrix} .$$

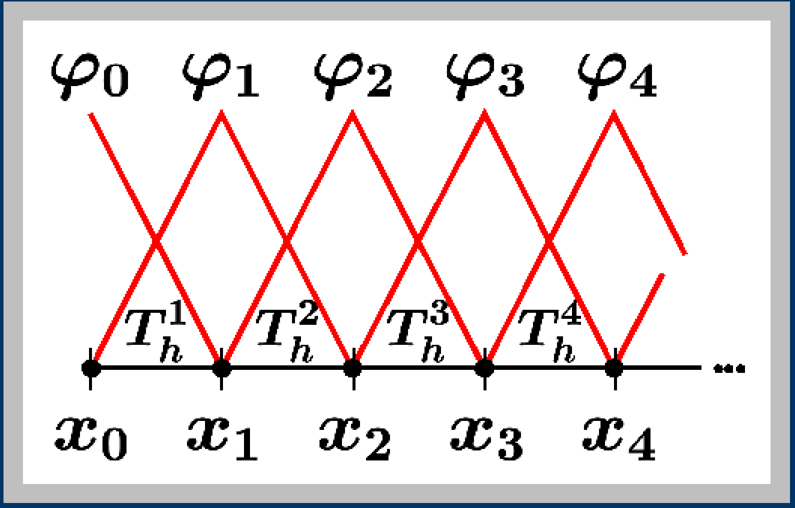
Evaluation (usually) by numerical quadrature.

Implementation

Assembly

The Idea...

Recall triangulation and basis functions:



Implementation

Assembly

...The Idea...

T_h^3 contribution to $\widetilde{A}_{h\,i\,j} = a(\varphi_i, \varphi_j) = \int_0^1 \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}\,dx$

$$\int_{T_h^3} \frac{d\varphi_{2\text{ or }3}}{dx}\frac{d\varphi_{\,2\text{ or }3}}{dx}\,dx = \underbrace{\begin{array}{c|cc} & 2 & 3 \\ \hline 2 & \frac{1}{h^3} & -\frac{1}{h^3} \\ 3 & -\frac{1}{h^3} & \frac{1}{h^3} \end{array}}_{\underline{A}^3}$$

	Column 1 of $\underline{A}^3$	Column 2 of $\underline{A}^3$
Row 1 of $\underline{A}^3$	Adds to $\widetilde{A}_{22}$	Adds to $\widetilde{A}_{23}$
Row 2 of $\underline{A}^3$	Adds to $\widetilde{A}_{32}$	Adds to $\widetilde{A}_{33}$

Implementation

Assembly

...The Idea...

$$
\underbrace{
\begin{array}{c|ccccc}
 & \text{C0} & \text{C1} & \text{C2} & \text{C3} & \text{C4} \\
\hline
\text{R0} & & & & & \\
\text{R1} & & & & & \\
\text{R2} & & & \frac{1}{h^3} & -\frac{1}{h^3} & \\
\text{R3} & & & -\frac{1}{h^3} & \frac{1}{h^3} & \\
\text{R4} & & & & & \\
\vdots & & & & &
\end{array}
}_{\underline{\widetilde{A}}_h \text{ with } T_h^3 \text{ accounted for } \ldots}
\qquad
\underline{A}^3 = \begin{pmatrix} \frac{1}{h^3} & -\frac{1}{h^3} \\ -\frac{1}{h^3} & \frac{1}{h^3} \end{pmatrix}
$$

Implementation

Assembly

...The Idea...

T_h^4 contribution to $\widetilde{A}_{h\,i\,j} = a(\varphi_i, \varphi_j) = \int_0^1 \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}\,dx$

$$\int_{T_h^4} \frac{d\varphi_{3 \text{ or } 4}}{dx}\frac{d\varphi_{3 \text{ or } 4}}{dx}\,dx = \underbrace{\begin{array}{c|cc} & 3 & 4 \\ \hline 3 & \frac{1}{h^4} & -\frac{1}{h^4} \\ 4 & -\frac{1}{h^4} & \frac{1}{h^4} \end{array}}_{\underline{A}^4}$$

	Column 1 of $\underline{A}^4$	Column 2 of $\underline{A}^4$
Row 1 of $\underline{A}^4$	Adds to $\widetilde{A}_{33}$	Adds to $\widetilde{A}_{34}$
Row 2 of $\underline{A}^4$	Adds to $\widetilde{A}_{43}$	Adds to $\widetilde{A}_{44}$

Implementation

Assembly

...The Idea...

$$
\underbrace{
\begin{array}{c}
\\ \text{R0} \\ \text{R1} \\ \text{R2} \\ \text{R3} \\ \text{R4} \\ \vdots
\end{array}
\begin{array}{c}
\begin{array}{cccccc}
\text{C0} & \text{C1} & \text{C2} & \text{C3} & \text{C4} & \ldots
\end{array}
\\
\left(
\begin{array}{cccccc}
 & & & & & \\
 & & & & & \\
 & & \frac{1}{h^3} & -\frac{1}{h^3} & & \\
 & & -\frac{1}{h^3} & \frac{1}{h^3} + \frac{1}{h^4} & -\frac{1}{h^4} & \\
 & & & -\frac{1}{h^4} & \frac{1}{h^4} & \\
 & & & & &
\end{array}
\right)
\end{array}
}_{\underline{\tilde{A}}_h \text{ with } T_h^3, T_h^4 \text{ accounted for } \ldots}
\qquad
\underline{A}^4 = \begin{pmatrix} \frac{1}{h^4} & -\frac{1}{h^4} \\ -\frac{1}{h^4} & \frac{1}{h^4} \end{pmatrix}
$$

Implementation

Assembly

...The Idea...

T_h^3 contribution to $\widetilde{F}_{h\,i} = \ell(\varphi_i) = \int_0^1 f\,\varphi_i\,dx$

$$\int_{T_h^3} f\,\varphi_{2 \text{ or } 3}\,dx = \begin{matrix} 2 \\ 3 \end{matrix} \underbrace{\begin{pmatrix} \dfrac{h^3}{2} \displaystyle\int_{-1}^{1} f\,\mathcal{H}_1\,d\zeta \\ \dfrac{h^3}{2} \displaystyle\int_{-1}^{1} f\,\mathcal{H}_2\,d\zeta \end{pmatrix}}_{\underline{F}^3}$$

Row 1 of $\underline{F}^3$	Adds to $\widetilde{F}_{h\,2}$
Row 2 of $\underline{F}^3$	Adds to $\widetilde{F}_{h\,3}$

Implementation

Assembly

...The Idea...

$$
\begin{matrix} R0 \\ R1 \\ R2 \\ R3 \\ R4 \\ \vdots \end{matrix}
\underbrace{\begin{pmatrix} \\ \\ F_1^3 \\ F_2^3 \\ \\ \\ \end{pmatrix}}_{\underline{\widetilde{F}}_h \text{ with } T_h^3 \text{ accounted for}}
$$

$$
\underline{F}^3 = \begin{pmatrix} F_1^3 \\ F_2^3 \end{pmatrix}
$$

Implementation

Assembly

...The Idea...

T_h^4 contribution to $\widetilde{\underline{F}}_{h\,i} = \ell(\varphi_i) = \int_0^1 f\,\varphi_i\,dx$

$$\int_{T_h^4} f\,\varphi_{3 \text{ or } 4}\;dx = \begin{matrix} 3 \\ 4 \end{matrix} \underbrace{\begin{pmatrix} \dfrac{h^4}{2} \displaystyle\int_{-1}^{1} f\,\mathcal{H}_1\,d\zeta \\ \dfrac{h^4}{2} \displaystyle\int_{-1}^{1} f\,\mathcal{H}_2\,d\zeta \end{pmatrix}}_{\underline{F}^4}$$

Row 1 of $\underline{F}^4$	Adds to $\widetilde{\underline{F}}_{h\,3}$
Row 2 of $\underline{F}^4$	Adds to $\widetilde{\underline{F}}_{h\,4}$

Implementation

Assembly

...The Idea

$$\begin{array}{l} R0 \\ R1 \\ R2 \\ R3 \\ R4 \\ \vdots \end{array} \underbrace{\begin{pmatrix} \\ \\ \boldsymbol{F}_1^3 & & \\ \boldsymbol{F}_2^3 & + & \boldsymbol{F}_1^4 \\ & & \boldsymbol{F}_2^4 \\ \\ \end{pmatrix}}$$

$$\underline{\boldsymbol{F}}^4 = \begin{pmatrix} \boldsymbol{F}_1^4 \\ \boldsymbol{F}_2^4 \end{pmatrix}$$

$\widetilde{\underline{\boldsymbol{F}}}_h$ with $\boldsymbol{T}_h^3, \boldsymbol{T}_h^4$ accounted for

Introduce local-to-global mapping:

$$\theta(k,\alpha)\colon \underbrace{\{1,\ldots,K\}}_{\text{element}} \times \underbrace{\{1,2\}}_{\text{local node number}} \to \underbrace{\{0,\ldots,n+1\}}_{\text{global node number}}$$

such that

x_α^k (local node α in element k) $=$

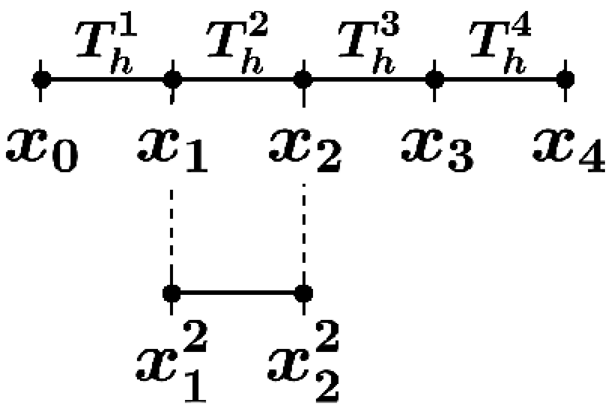
$x_{\theta(k,\alpha)}$ (global node $\theta(k,\alpha)$).

Implementation

Assembly

...The Algorithm...

Example: $K = 4$



$\theta(k, \alpha)$

α \ k	1	2	3	4
1	0	1	2	3
2	1	2	3	4

Implementation

Assembly

...The Algorithm...

Procedure for $\underline{\widetilde{A}}_h$:

zero $\underline{\widetilde{A}}_h$;

$\{$for $k = 1, \ldots, K$

$\{$for $\alpha = 1, 2$

$i = \theta(k, \alpha)$;

$\{$for $\beta = 1, 2$

$j = \theta(k, \beta)$;

$\widetilde{A}_{h\,i\,j} = \widetilde{A}_{h\,i\,j} + A^k_{\alpha\beta}$; $\}$ $\}$ $\}$

Implementation

Assembly

...The Algorithm

Procedure for $\underline{\widetilde{F}}_h$:

zero $\underline{\widetilde{F}}_h$;

$\{$for $k = 1, \ldots, K$

$\{$for $\alpha = 1, 2$

$i = \theta(k, \alpha)$;

$$\widetilde{F}_{h\,i} = \widetilde{F}_{h\,i} + F_\alpha^k \;;\} \,\}$$

Implementation

Boundary Conditions

Point of Departure

$$\underbrace{\frac{1}{h}\begin{pmatrix} 1 & -1 & & & & & 0 \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & & \ddots & & & \\ & 0 & & & -1 & 2 & -1 \\ & & & & & -1 & 1 \end{pmatrix}}_{\underline{\widetilde{A}}_h} \underbrace{\begin{pmatrix} \tilde{u}_{h\,0} \\ \tilde{u}_{h\,1} \\ \cdot \\ \cdot \\ \cdot \\ \tilde{u}_{h\,n} \\ \tilde{u}_{h\,n+1} \end{pmatrix}}_{\underline{\tilde{u}}_h} = \underbrace{\begin{pmatrix} \widetilde{F}_{h\,0} \\ \widetilde{F}_{h\,1} \\ \cdot \\ \cdot \\ \cdot \\ \widetilde{F}_{h\,n} \\ \widetilde{F}_{h\,n+1} \end{pmatrix}}_{\underline{\widetilde{F}}_h}$$

Implementation

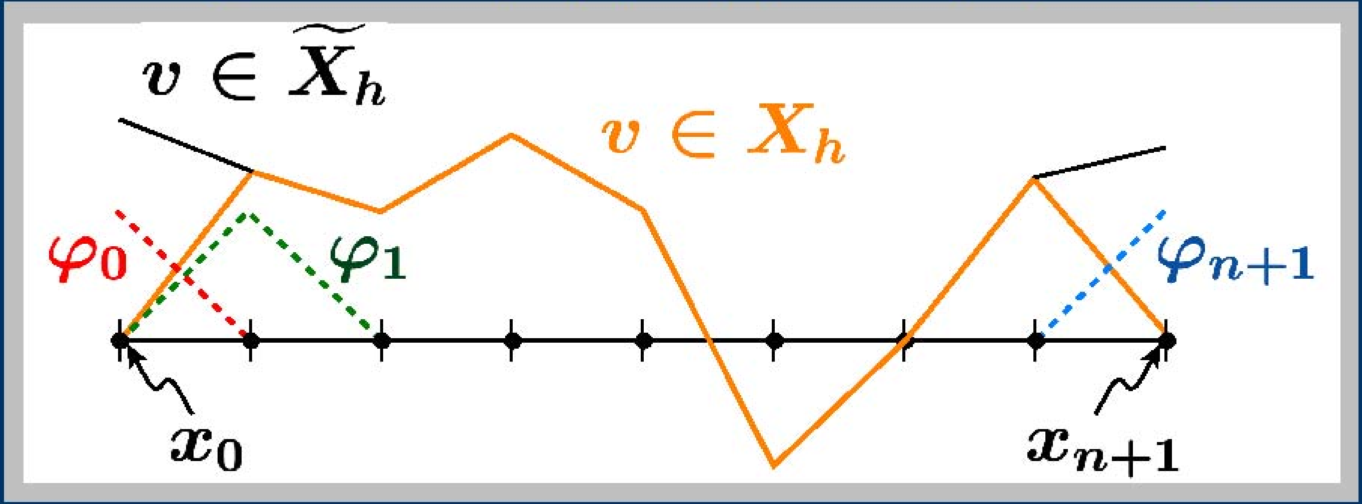
Boundary Conditions

Homogeneous Dirichlet...

$u_h \in X_h$ such that $a(u_h, v) = \ell(v), \quad \forall\, v \in X_h$:

$$X_h = \{v \in X \mid v|_{T_h} \in \mathbb{P}_1(T_h),\ \forall\, T_h \in \mathcal{T}_h\}\ ;$$

$$X = \{v \in H^1(\Omega) \mid v(0) = v(1) = 0\}\ .$$



Implementation

Boundary Conditions

...Homogeneous Dirichlet...

Explicit Elimination

$\boldsymbol{X}_h \Rightarrow \ \varphi_0, \varphi_{n+1}$ not admissible variations, so

REMOVE $\boldsymbol{R0}$ and $\boldsymbol{Rn+1}$ from $\underline{\widetilde{A}}_h$;

$\tilde{u}_{h\,0} = \tilde{u}_{h\,n+1} = \mathbf{0}$, so

REMOVE $\boldsymbol{C0}$ and $\boldsymbol{Cn+1}$ from $\underline{\widetilde{A}}_h$.

Recover $\underline{A}_h\, \underline{u}_h = \underline{F}_h$

Implementation

Boundary Conditions

...Homogeneous Dirichlet

Big-Number Approach penalty

Place $1/\varepsilon$ $(\varepsilon \ll 1)$ on entries $\widetilde{A}_{h\,0\,0}$ and $\widetilde{A}_{h\,n+1\,n+1}$.

Place 0 on entries $\widetilde{F}_{h\,0}$ and $\widetilde{F}_{h\,n+1}$.

This replaces $R0$ and $Rn+1$ with

$$\tilde{u}_{h\,0} \cong 0,\ \tilde{u}_{h\,n+1} \cong 0$$

in an "easy," symmetric way.

Implementation

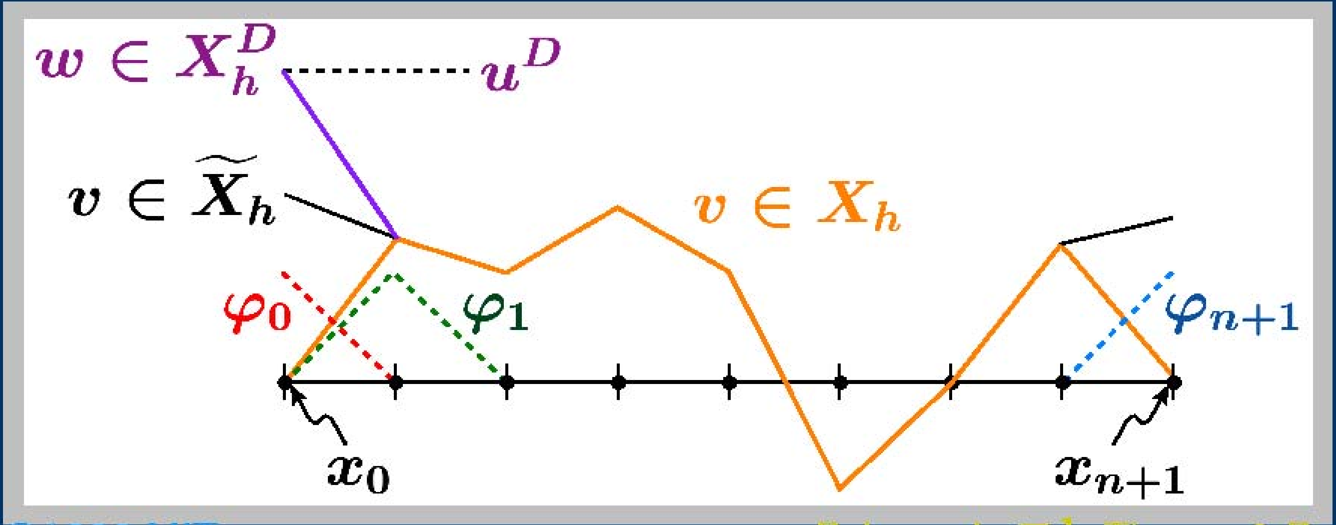
Boundary Conditions

Inhomogeneous Dirichlet...

$u_h \in X_h^D$ such that $a(u_h, v) = \ell(v), \quad \forall\, v \in X_h$:

X_h requires $v(0) = v(1) = 0$;

X_h^D requires $w(0) = u^D,\ w(1) = 0$.



Implementation

Boundary Conditions

...Inhomogeneous Dirichlet...

Explicit Elimination . . .

$X_h \Rightarrow \; \varphi_0, \varphi_{n+1}$ not admissible variations, so

REMOVE $R0$ and $Rn+1$ from $\underline{\widetilde{A}}_h$;

$X_h^D \Rightarrow \; \tilde{u}_{h\,0} = u^D, \; \tilde{u}_{h\,n+1} = 0$, so

MOVE $-u^D\,C0 - 0\,Cn + 1$ to $\underline{\widetilde{F}}_h$.

Implementation

Boundary Conditions

...Inhomogeneous Dirichlet...

... Explicit Elimination

$$\underbrace{\frac{1}{h}\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & 0 \\ & -1 & 2 & -1 & & \\ & & & \ddots & & \\ & 0 & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}}_{A_h} \begin{pmatrix} \tilde{u}_{h\,1} \\ \vdots \\ \tilde{u}_{h\,n} \end{pmatrix} = \begin{pmatrix} \widetilde{F}_{h\,1} - u^D \times (-\frac{1}{h}) \\ \vdots \\ \widetilde{F}_{h\,n} \end{pmatrix}$$

Implementation

Boundary Conditions

...Inhomogeneous Dirichlet

Big-Number Approach

E7

Place $1/\varepsilon$ $(\varepsilon \ll 1)$ on entries $\widetilde{A}_{h\,0\,0}$ and $\widetilde{A}_{h\,n+1\,n+1}$.

Place $(1/\varepsilon)\,u^D$ on entry $\widetilde{F}_{h\,0}$.

Place 0 on entry $\widetilde{F}_{h\,n+1}$.

This replaces $R0$ and $Rn+1$ with

$$\tilde{u}_{h\,0} \cong u^D,\ \tilde{u}_{h\,n+1} \cong 0.$$

Implementation

Quadrature

Question...

How do we evaluate

$$F_\alpha^k = \frac{h^k}{2} \int_{-1}^{1} \underbrace{f\left(x_1^k + \frac{(1+\zeta)}{2}\, h^k\right)}_{f^k(\zeta)} \mathcal{H}_\alpha(\zeta)\, d\zeta$$

for general f?

N11

Implementation

Quadrature

...Question

Approaches

- “Analytical” Integration
- Symbolic Integration
- Gauss Quadrature $\leftarrow$
- Integration by Interpolation

N12

Implementation

Quadrature

Gauss Quadrature...

Approximate

$$F_\alpha^k = \frac{h^k}{2} \int_{-1}^{1} f^k(\zeta)\, \mathcal{H}_\alpha(\zeta)\, d\zeta$$

$$\approx \frac{h^k}{2} \sum_{q=1}^{N_q} \rho_q\, f^k(z_q)\, \mathcal{H}_\alpha(z_q):$$

ρ_q: Gauss-Legendre quadrature weights

z_q: Gauss-Legendre quadrature points.

Implementation

Quadrature

Gauss Quadrature...

The $\rho_q, z_q,\ q = 1, \ldots, N_q$ are chosen so as

to integrate exactly all $g \in \mathbb{P}_{2N_q-1}((-1,1))$. **N13**

To conserve "ideal" convergence rates,

require $N_q \geq 1$ ($\geq p$ for $\mathbb{P}_p$ elements).