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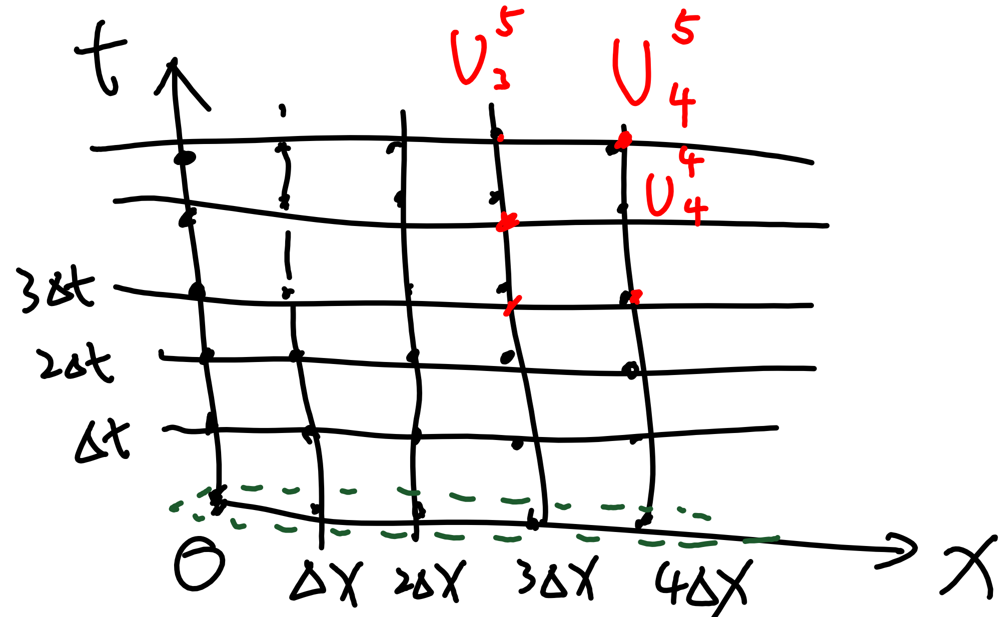
Finite Difference approximation of partial derivatives

$U(x,t)$

$\frac{\partial U}{\partial x}$ $\frac{\partial U}{\partial t}$



$$\left.\frac{\partial U}{\partial x}\right|_i^n = \frac{U_{i+1}^n - U_i^n}{\Delta x}$$

Forward in space

$$\left.\frac{\partial U}{\partial x}\right|_i^n = \frac{U_i^n - U_{i-1}^n}{\Delta x}$$

Backward in space

$$\left.\frac{\partial U}{\partial x}\right|_i^n = \frac{U_{i+1}^n - U_{i-1}^n}{2\Delta x}$$

Central in space

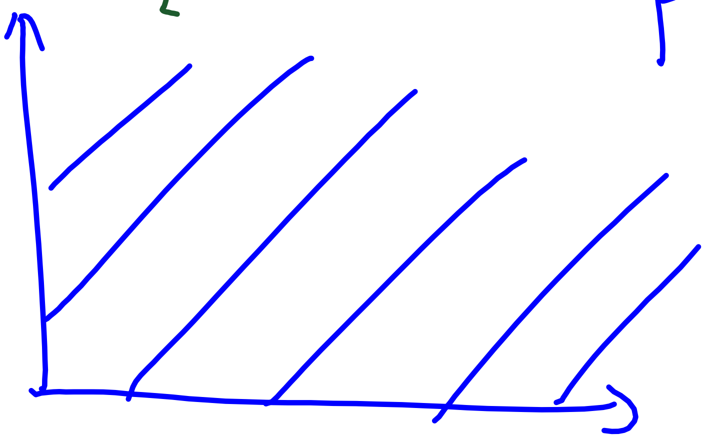
$$\left.\frac{\partial U}{\partial t}\right|_i^n = \frac{U_i^{n+1} - U_i^n}{\Delta t}$$

Forward Euler

$$\frac{\partial U}{\partial t} + \underset{\substack{\| \\ 1}}{c}\frac{\partial U}{\partial x} = 0$$

$U(0) = U(1)$

periodic BC



Finite Difference approximation of second order spatial derivative

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}$$

$$\boxed{\left.\frac{\partial^2 U}{\partial x^2}\right|_i^n \approx \frac{U_{i+1}^n \cancel{- 2U_i^n} + U_{i-1}^n}{\Delta x^2}} + O(\Delta x^4)$$

$$U_{i+1}^n = \cancel{U_i^n} + \cancel{\Delta x \left.\frac{\partial U}{\partial x}\right|_i^n} + \frac{\Delta x^2}{2}\left.\frac{\partial^2 U}{\partial x^2}\right|_i^n + \cancel{\frac{\Delta x^3}{6}\frac{\partial^3 U}{\partial x^3}}$$

$$U_{i-1}^n = \cancel{U_i^n} - \cancel{\Delta x \left.\frac{\partial U}{\partial x}\right|_i^n} + \frac{\Delta x^2}{2}\left.\frac{\partial^2 U}{\partial x^2}\right|_i^n - \cancel{\frac{\Delta x^3}{6}\frac{\partial^3 U}{\partial x^3}} + O(\Delta x^4)$$

$$\frac{\partial^2 U}{\partial x^2} - \frac{U_{i+1} - 2U_i + U_{i-1}}{\Delta x^2} \sim O(\Delta x^2)$$

Finite Difference Simulation

Finite Difference in Matrix Form

Central Difference

$$\left.\frac{\partial u}{\partial x}\right|_i = \frac{u_{i+1} - u_{i-1}}{2\Delta x}$$

$u_1\ u_2 \cdot \quad u_N$ we have

$$\begin{pmatrix} \left.\frac{\partial u}{\partial x}\right|_1 \\ \left.\frac{\partial u}{\partial x}\right|_2 \\ \vdots \\ \left.\frac{\partial u}{\partial x}\right|_n \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2\Delta x} & 0 & - & - & - & -\frac{1}{2\Delta x} \\ -\frac{1}{2\Delta x} & 0 & \frac{1}{2\Delta x} & 0 & & - & 0 \\ & -\frac{1}{2\Delta x} & 0 & \frac{1}{2\Delta x} & & & \\ & & -\frac{1}{2\Delta x} & 0 & \frac{1}{2\Delta x} & & \\ & & & \ddots & \ddots & \ddots & \frac{1}{2\Delta x} \\ \frac{1}{2\Delta x} & & & & & -\frac{1}{2\Delta x} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

Finite Difference in Matrix Form

Backward - Space ~~Upwind~~ Difference

$$\left.\frac{\partial U}{\partial x}\right|_i = \frac{U_i - U_{i-1}}{\Delta x}$$

$$\begin{bmatrix} \left.\frac{\partial U}{\partial x}\right|_1 \\ \left.\frac{\partial U}{\partial x}\right|_2 \\ \vdots \\ \left.\frac{\partial U}{\partial x}\right|_n \end{bmatrix} = \begin{bmatrix} \frac{1}{\Delta x} & 0 & & 0 & -\frac{1}{\Delta x} \\ -\frac{1}{\Delta x} & \frac{1}{\Delta x} & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & -\frac{1}{\Delta x} & \frac{1}{\Delta x} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix}$$

Finite Difference in Matrix Form

Backward -in-space

~~Upwind~~ Difference with BC

$$u(x=0,\ t) = 1$$

Back-in-space

$$\left.\frac{\partial U}{\partial x}\right|_1^n = \frac{U_1 - \overset{1}{\overset{\shortparallel}{U_0}}}{\Delta x}$$

$$= \frac{U_1}{\Delta x} - \frac{1}{\Delta x}$$

$$\begin{bmatrix} \frac{\partial u}{\partial x}\Big|_1 \\ \frac{\partial u}{\partial x}\Big|_2 \\ \vdots \\ \vdots \\ \frac{\partial u}{\partial x}\Big|_n \end{bmatrix} = \begin{bmatrix} \frac{1}{\Delta x} & \frac{1}{\Delta x} & & & \\ -\frac{1}{\Delta x} & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ & & & -\frac{1}{\Delta x} & \frac{1}{\Delta x} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ \vdots \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} -\frac{1}{\Delta x} \\ 0 \\ \vdots \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

Finite Difference in Matrix Form

Second Order Derivative

$$\left.\frac{\partial^2 U}{\partial x^2}\right|_i = \frac{U_{i+1}}{\Delta x^2} - \frac{2U_i}{\Delta x^2} + \frac{U_{i-1}}{\Delta x^2}$$

$$\begin{pmatrix} \left.\frac{\partial^2 U}{\partial x^2}\right|_1 \\ \vdots \\ \\ \\ \left.\frac{\partial^2 U}{\partial x^2}\right|_n \end{pmatrix} = \begin{pmatrix} \frac{-2}{\Delta x^2} & \frac{1}{\Delta x^2} & & & \frac{1}{\Delta x^2} \\ \frac{1}{\Delta x^2} & \frac{-2}{\Delta x^2} & \frac{1}{\Delta x^2} & & \\ & \frac{1}{\Delta x^2} & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{1}{\Delta x^2} \\ \frac{1}{\Delta x^2} & & & \frac{1}{\Delta x^2} & -\frac{2}{\Delta x^2} \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ \\ U_n \end{pmatrix}$$

Finite Difference in Matrix Form
Application to Backward Euler

Finite Difference in Matrix Form
Application to Trapezoidal Rule

Finite Difference in Matrix Form
Application to 1D Poisson Equation

16.90 Computational Methods in Aerospace Engineering
Spring 2014