16.90 Lecture 16 - 18:
The Finite Element Method for 1D Diffusion

Today's Topics

1. Key ideas
2. Brief history
3. 1D linear elements
4. The nodal basis
5. FEM derivation and implementation for 1D diffusion

1. Key Ideas

1. Key ideas: Based on method of weighted residuals
   - Discretize domain into small cells (elements)
   - Approximate the solution in each element, eg. with polynomial functions
   - Evaluate weighted residuals for each element
     → system of equations, solved to determine weighting coefficients

4. The Nodal Basis

4. Nodal basis functions for 1D Linear elements

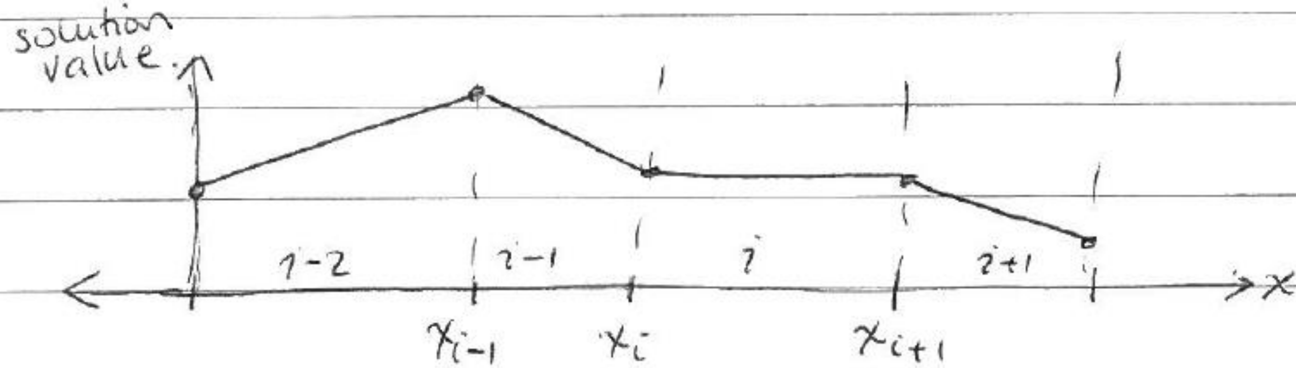
- Need N+1 basis functions

$$\phi_1(x),\ \phi_2(x),\ \ldots,\ \phi_{N+1}(x)$$

For nodal basis,
- Solution is described by values of function at the N+1 nodes

ie. $$\tilde{T}(x) = \sum_{i=1}^{N+1} a_i\,\phi_i(x)$$

$$= \sum_{i=1}^{N+1} \tilde{T}(x_i)\,\phi_i(x)$$



- In each element have 2 dof.
  eg. values at two endpoints (ie. nodes)
  or value at one endpoint and slope.

- For N elements, have N+1 dof

e.g.

$N=5$, 6 dof

5 elements, 2 dof each, but 4 continuity cond^{n}s $\Bigg\}$ ~~# dof =~~ $5\times2-4 = 6$.

4. The Nodal Basis

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$$\text{ie.}\quad \tilde{T}(x) = \sum_{i=1}^{N+1} a_i\, \phi_i(x)$$

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For nodal basis,
- Solution is described by values of function at the N+1 nodes

i.e. $$\tilde{T}(x) = \sum_{i=1}^{N+1} a_i \phi_i(x)$$

$$= \sum_{i=1}^{N+1} \tilde{T}(x_i) \phi_i(x)$$

• Apply the expansion at node j:

$$\tilde{T}(x_j) = \sum_{i=1}^{N+1} a_i \, \phi_i(x_j)$$

value of basis function i at node x_j

$$\tilde{T}(x_j) = \sum_{i=1}^{N+1} \tilde{T}(x_i)\, \phi_i(x_j)$$

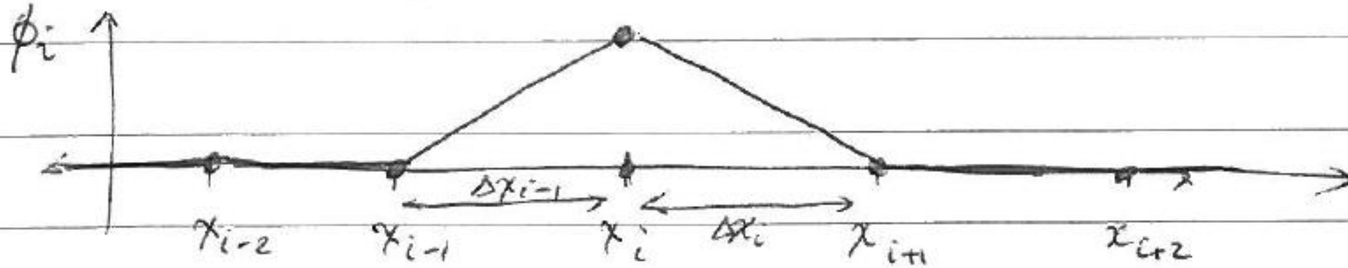
$$\tilde{T}(x_j) = \sum_{i=1}^{j-1} \tilde{T}(x_i)\phi_i(x_j) + \tilde{T}(x_j)\phi_j(x_j) + \sum_{i=j+1}^{N+1} \tilde{T}(x_i)\phi_i(x_j)$$

• This equation must be satisfied for any x_j

$$\Rightarrow \quad \phi_i(x_j) = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{if } i\neq j \end{cases}$$

- What are the values of $\phi_i(x)$ for other points not at the nodes?

→ Since the solutions vary linearly within each element, the basis functions must also vary linearly within each element.



$$\phi_i(x) = \begin{cases} 0 , & x < x_{i-1} \\ \dfrac{x - x_{i-1}}{\Delta x_{i-1}} , & x_{i-1} < x < x_i \\ \dfrac{x_{i+1} - x}{\Delta x_i} , & x_i < x < x_{i+1} \\ 0 , & x > x_{i+1} \end{cases}$$

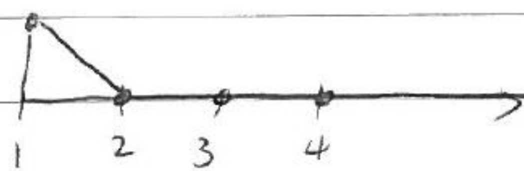
• The solution at any point x is represented by a linear combination of these basis functions

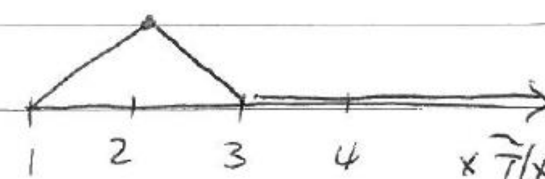
$$\tilde{T}(x) = \sum_{i=1}^{N+1} \tilde{T}(x_i)\,\phi_i(x)$$

approx. sol^n at x

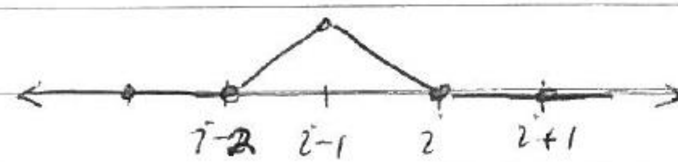
nodal values of $\tilde{T}$

value of basis function i at x.

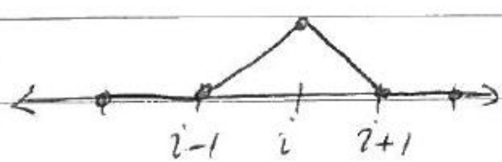
$\tilde{T}(x) =$ 

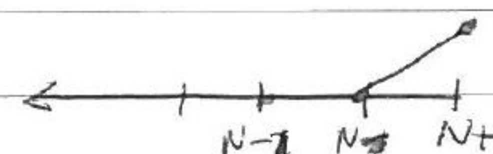
$\times\, \tilde{T}(x_1)$ + 

$\times\, \tilde{T}(x_2)$

+ . . . + 

$\times\, \tilde{T}(x_{i-1})$ +



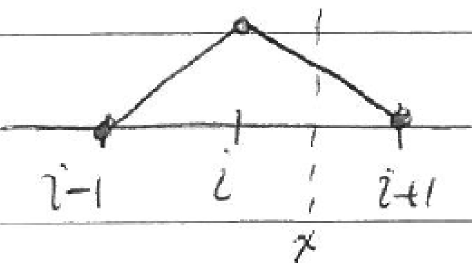
$\times\, \tilde{T}(x_i)$ + . . . + 

$\times\, \tilde{T}(x_{N+1})$

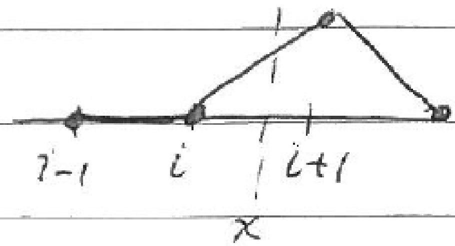
• Note, for a point x in element i,

$$x_i < x < x_{i+1}$$

only ϕ_i and ϕ_{i+1} will be non-zero.



$\times\ \tilde{T}(x_i)$ +



$\times\ \tilde{T}(x_{i+1})$

Solution at x is a linear combination of solutions at x_i and x_{i+1}.

MIT 16.90 Spring 2014
The Finite Element Method for 1D Diffusion
In-class exercise, April 9 2014

Recall our model problem of steady heat diffusion in a rod. We model this problem as a one-dimensional PDE for the temperature, T:

$$(kT_x)_x = -f, \tag{1}$$

where $k = 1$ is the thermal conductivity of the material and $f(x) = 50e^x$ is the heat source per unit length. The physical domain for the problem is from $x = -1$ to $x = 1$. The boundary conditions specify that the temperatures at the ends of the rod are to be maintained at $T(\pm 1) = 100$.

The exact solution for this problem is

$$T = -50e^x + 50x\sinh(1) + 100 + 50\cosh(1). \tag{2}$$

1 The solution approximation

Let's discretize the domain into N_e elements. What is the approximation of the solution using the nodal basis?

2 Derive the weighted residual

First, write down the expression for the residual:

Now, let's derive the weighted residual:

3 Evaluate the entries in the stiffness matrix

Let's evaluate the term

$$\int_{-L/2}^{L/2} \phi_{j_x} \, k\tilde{T}_x \, dx$$

4 Form the stiffness matrix

Sketch out the form of the stiffness matrix for the case $N_e = 5$:

5 Evaluate the forcing term

Evaluate the term in the residual that involves the heat source $f(x)$.

6 Implement the code

Download the shell code `fem_dif1d_shell.m` from Course Website. Hint: The most efficient way to form the stiffness matrix is to loop over elements. If we loop over elements, what is the pattern we need to fill in on each loop?