

Moments of the Boltzmann Equation

As was already discussed, finding solutions to Boltzmann's equation can be formidably difficult, even in the simplest of cases. An important manipulation can be achieved by taking moments (or averages) of pertinent quantities and try to recover fluid-type conservation equations. On their own right, these fluid equations (like the Navier-Stokes equations) are also very difficult to solve analytically, unless special cases are treated, but at least they provide us with a useful physical picture of the system behavior. In what follows we will ignore the effects of inelastic collisions.

As before, we write Boltzmann's equation

$$\frac{\partial f_s}{\partial t} + \vec{w} \cdot \nabla f_s + \frac{\vec{F}_s}{m_s} \cdot \nabla_w f_s = \sum_r \int_{w_1} \int_{\Omega} (f'_s f'_r - f_s f_r) g \sigma_{rs} d\Omega d^3 w_1$$

We define a general function $\phi = \phi(\vec{x}, \vec{w}, t)$, multiply Boltzmann's equation with it and then integrate over all velocities \vec{w} . One by one, the terms of Boltzmann's equation will result in

$$1. \int \frac{\partial f_s}{\partial t} \phi d^3 w = \int \left[\frac{\partial}{\partial t} (f_s \phi) - f_s \frac{\partial \phi}{\partial t} \right] d^3 w$$

Recalling the definition of the average of a quantity: $\langle \phi \rangle = \frac{1}{n} \int \phi f d^3 w$, and exchanging the integral and derivative symbols, we obtain

$$\int \frac{\partial f_s}{\partial t} \phi d^3 w = \frac{\partial}{\partial t} (n_s \langle \phi \rangle_s) - n_s \left\langle \frac{\partial \phi}{\partial t} \right\rangle_s$$

$$2. \int \phi \vec{w} \cdot \nabla f_s d^3 w = \int [\nabla \cdot (\phi \vec{w} f_s) - \phi f_s \nabla \cdot \vec{w} - f_s \vec{w} \cdot \nabla \phi] d^3 w$$

Since $\vec{w} \neq \vec{w}(\vec{x})$, the second term in the RHS vanishes. Then we have

$$\int \phi \vec{w} \cdot \nabla f_s d^3 w = \nabla \cdot n_s \langle \phi \vec{w} \rangle_s - n_s \langle \vec{w} \cdot \nabla \phi \rangle_s$$

$$3. \int \phi \frac{\vec{F}_s}{m_s} \cdot \nabla_w f_s d^3 w = \frac{1}{m_s} \int [\nabla_w \cdot (\phi \vec{F}_s f_s) - \phi f_s \nabla_w \cdot \vec{F}_s - f_s \vec{F}_s \cdot \nabla_w \phi] d^3 w$$

From Liouville's theorem, \vec{F} is a conservative and/or magnetic force, therefore the second term in the RHS vanishes. The first term vanishes as well. To see why, we use the divergence theorem and write it as a surface integral in velocity space

$$\int \nabla_w \cdot (\phi \vec{F}_s f_s) d^3 w = \oint \phi f_s \vec{F}_s \cdot dS_w. \text{ The volume integral is evaluated for every}$$

possible value in velocity space, therefore the surface boundary is located at infinity, where there are no particles at all (the distribution function vanishes there).

$$\int \phi \frac{\vec{F}_s}{m_s} \cdot \nabla_w f_s d^3 w = -n_s \left\langle \frac{\vec{F}_s}{m_s} \cdot \nabla_w \phi \right\rangle_s$$

$$4. \int \phi \left(\frac{\partial f}{\partial t} \right)_{coll} d^3 w = \sum_r \int \int \int \int \phi (f'_s f'_{r_1} - f_s f_{r_1}) g \sigma_{rs} d\Omega d^3 w d^3 w_1$$

We can split this expression into two integrals. In the first one we exchange the coordinates after the collision with those before it ($\vec{w} \leftrightarrow \vec{w}'$). And since (as seen before) the Jacobian of the transformation is unity ($d^3 w'_1 d^3 w' = d^3 w_1 d^3 w$) and the magnitude of the relative velocity does not change, then

$$\int \phi \left(\frac{\partial f}{\partial t} \right)_{coll} d^3 w = \sum_r \int \int \int \int (\phi' - \phi) f_s f_{r_1} g \sigma_{rs} d\Omega d^3 w d^3 w_1$$

Adding all terms we have the general moment equation

$$\begin{aligned} \frac{\partial}{\partial t} (n_s \langle \phi \rangle_s) - n_s \left\langle \frac{\partial \phi}{\partial t} \right\rangle_s + \nabla \cdot n_s \langle \phi \vec{w} \rangle_s - n_s \langle \vec{w} \cdot \nabla \phi \rangle_s - n_s \left\langle \frac{\vec{F}_s}{m_s} \cdot \nabla_w \phi \right\rangle_s = \\ = \sum_r \int \int \int \int (\phi' - \phi) f_s f_{r_1} g \sigma_{rs} d\Omega d^3 w d^3 w_1 \end{aligned}$$

Now let us obtain the moments for different values of the function $\phi = \phi(\vec{x}, \vec{w}, t)$

a. $\phi = 1$ (Mass)

All the terms where derivatives of ϕ appear will vanish, leading to

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \vec{u}_s) = 0 \quad \text{or} \quad \frac{\partial \rho_s}{\partial t} + \nabla \cdot (\rho_s \vec{u}_s) = 0$$

where the expression in the right was obtained after multiplying both sides with the species mass m_s . Finally, adding contributions of all species: $\sum_s \rho_s = \rho$ (fluid density) and $\sum_s \rho_s \vec{u}_s = \rho \vec{u}$ (fluid mass flux), we write the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \quad \text{where} \quad \vec{u} = \frac{\sum_s \rho_s \vec{u}_s}{\sum_s \rho_s}$$

b. $\phi = m_s \vec{w}$ (Momentum)

Note that $n_s \langle \phi \rangle_s = \rho_s \vec{u}_s$. Also, since \vec{w} has no explicit dependence on t or \vec{x} , then

$$n_s \left\langle \frac{\partial \phi}{\partial t} \right\rangle_s = 0 \quad \text{and} \quad n_s \langle \vec{w} \cdot \nabla \phi \rangle_s = 0$$

From the definition of random velocity for species s , $\vec{w} = \vec{u}_s + \vec{c}_s$, and as $\langle \vec{c}_s \rangle = 0$

$$n_s \langle \phi \vec{w} \rangle_s = \rho_s \langle \vec{w} \vec{w} \rangle_s = \rho_s \langle (\vec{u}_s + \vec{c}_s)(\vec{u}_s + \vec{c}_s) \rangle = \rho_s (\vec{u}_s \vec{u}_s + \langle \vec{c}_s \vec{c}_s \rangle_s) = \rho_s \vec{u}_s \vec{u}_s + \vec{P}'_s$$

where \vec{P}'_s is the partial pressure of species s across planes moving at \vec{u}_s . The prime is to indicate that the quantity (in this case the pressure) is taken with respect to the random velocity of species s .

Now, taking the electromagnetic force $\vec{F}_s = q_s (\vec{E} + \vec{w} \times \vec{B})$, we have

$$n_s \left\langle \frac{\vec{F}}{m_s} \cdot \nabla_w \phi \right\rangle = n_s q_s \left\langle (\vec{E} + \vec{w} \times \vec{B}) \cdot \vec{I} \right\rangle_s = n_s q_s (\vec{E} + \vec{u}_s \times \vec{B})$$

For the collision part, we write $\phi' - \phi = m_s (\vec{w}' - \vec{w})$. In terms of the center of mass and relative velocity, $\vec{g} = \vec{w}_1 - \vec{w}$

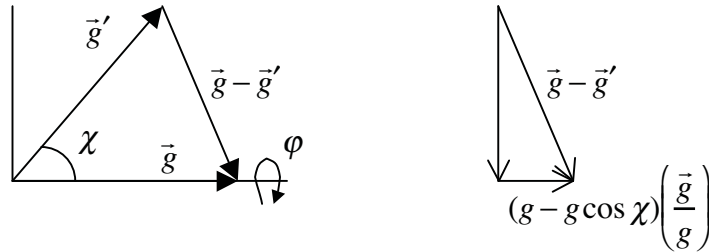
$$\vec{w} = \vec{G} - \frac{m_r}{m_r + m_s} \vec{g} \quad \text{and} \quad \vec{w}' = \vec{G} - \frac{m_r}{m_r + m_s} \vec{g}'$$

then

$$m_s (\vec{w}' - \vec{w}) = \frac{m_s m_r}{m_r + m_s} (\vec{g} - \vec{g}') = \mu_{sr} (\vec{g} - \vec{g}')$$

and the collision integral can be written as

$$\vec{M}_{rs} = \mu_{rs} \int_{w_1} \int_w f_s f_r g d^3 w d^3 w_1 \int_{\Omega} \sigma_{rs} (\vec{g} - \vec{g}') d\Omega$$



We observe from the diagram above, that the component of the $\vec{g} - \vec{g}'$ vector perpendicular to \vec{g} will cancel out after integrating over all angles ϕ . What survives is just the component parallel to \vec{g} . Given this, and the fact that the magnitude of the relative velocity is invariant, we write

$$\vec{M}_{rs} = \mu_{rs} \int_{w_1} \int_w f_s f_r g d^3 w d^3 w_1 \int_{\Omega} \sigma_{rs} \vec{g} (1 - \cos \chi) d\Omega$$

Separating the angular contribution we obtain the momentum transfer cross section

$$Q_{rs}^{(1)}(g) = \int_{\Omega} \sigma_{rs}(\chi, g)(1 - \cos \chi) d\Omega$$

The collision integral becomes

$$\vec{M}_{rs} = \mu_{rs} \int_{w_1} \int_w f_s f_{r_1} g \vec{g} Q_{rs}^{(1)}(g) d^3 w d^3 w_1$$

which is the rate of momentum transfer from species s to species r per unit volume due to collisions. Adding up all terms,

$$\frac{\partial}{\partial t}(\rho_s \vec{u}_s) + \nabla \cdot (\rho_s \vec{u}_s \vec{u}_s) + \nabla \cdot \vec{P}'_s - n_s q_s (\vec{E} + \vec{u}_s \times \vec{B}) = \sum_r \vec{M}_{rs}$$

Since we averaged absolute momentum, this is the Eulerian form, with $\nabla \cdot$ (momentum flux).

- c. $\phi = m_s(\vec{w} - \vec{u}_s)$ (Momentum with deviations from the species s mean velocity)
 Instead of repeating the process all over again, we recognize that such function can be separated in two additive terms, the first one $\phi_1 = m_s \vec{w}$ will lead to the same momentum equation found in (b) while the second $\phi_2 = m_s \vec{u}_s$ is proportional to the result in (a) since \vec{u}_s is already an averaged quantity. Using the result in (a) for $\phi_2 = m_s \vec{u}_s$ we obtain

$$m_s \vec{u}_s \left[\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \vec{u}_s) \right]$$

then we subtract this from the result in (b)

$$\frac{\partial}{\partial t}(\rho_s \vec{u}_s) + \nabla \cdot (\rho_s \vec{u}_s \vec{u}_s) + \nabla \cdot \vec{P}'_s - n_s q_s (\vec{E} + \vec{u}_s \times \vec{B}) - m_s \vec{u}_s \left[\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \vec{u}_s) \right] = \sum_r \vec{M}_{rs}$$

For the second term in this expression, using tensorial notation

$$\nabla \cdot (\rho_s \vec{u}_s \vec{u}_s) = \frac{\partial}{\partial x_i} (\rho_s u_i u_j) = \rho_s u_i \frac{\partial u_j}{\partial x_i} + u_j \frac{\partial}{\partial x_i} (\rho_s u_i) = \rho_s (\vec{u}_s \cdot \nabla) \vec{u}_s + \vec{u}_s \nabla \cdot (\rho_s \vec{u}_s)$$

After regrouping we obtain

$$\rho_s \left[\frac{\partial}{\partial t} \vec{u}_s + (\vec{u}_s \cdot \nabla) \vec{u}_s \right] + \nabla \cdot \vec{P}'_s - n_s q_s (\vec{E} + \vec{u}_s \times \vec{B}) = \sum_r \vec{M}_{rs}$$

or

$$\rho_s \frac{D_s \bar{u}_s}{Dt} + \nabla \cdot \bar{\bar{P}}'_s - n_s q_s (\bar{E} + \bar{u}_s \times \bar{B}) = \sum_r \bar{M}_{rs}$$

This is the Lagrangian form of the momentum equation for species s . Note that since we averaged deviations from \bar{u}_s , we get substantial derivatives at \bar{u}_s .

- d. $\phi = m_s (\bar{w} - \bar{u})$ (Momentum with deviations from the fluid mean velocity)
 Start by noting that $n_s \langle \phi \rangle_s = \rho_s (\bar{u}_s - \bar{u}) = \rho_s \bar{V}_s$, where \bar{V}_s is the diffusion velocity of species s . Then, one by one, the terms of the moment equation will be

$$\frac{\partial}{\partial t} (n_s \langle \phi \rangle_s) = \frac{\partial}{\partial t} (\rho_s \bar{V}_s)$$

and

$$n_s \left\langle \frac{\partial \phi}{\partial t} \right\rangle_s = -\rho_s \frac{\partial \bar{u}}{\partial t} \quad (\text{remember that } \bar{w} \text{ does not depend explicitly on } t).$$

Also

$$\nabla \cdot n_s \langle \phi \bar{w} \rangle_s = \nabla \cdot (\rho_s \langle (\bar{w} - \bar{u}) \bar{w} \rangle_s)$$

Defining the fluid random velocity $\bar{c} = \bar{w} - \bar{u}$, and noting that $\langle \bar{c} \rangle_s = \bar{u}_s - \bar{u} = \bar{V}_s$, the last term can be written as

$$\nabla \cdot n_s \langle \phi \bar{w} \rangle_s = \nabla \cdot (\rho_s \langle \bar{c} (\bar{c} + \bar{u}) \rangle_s) = \nabla \cdot (\rho_s \langle \bar{c} \bar{c} \rangle_s + \rho_s \langle \bar{c} \bar{u} \rangle_s) = \nabla \cdot (\bar{\bar{P}}_s + \rho_s \bar{V}_s \bar{u})$$

after defining the pressure tensor $\bar{\bar{P}}_s = \rho_s \langle \bar{c} \bar{c} \rangle_s$, which can be written in terms of the partial pressure $\bar{\bar{P}}'_s$. To see this, we use the definition of the random velocity of species s , $\bar{c}_s = \bar{w} - \bar{u}_s$, so that $\bar{c} - \bar{c}_s = \bar{u}_s - \bar{u} = \bar{V}_s$ and $\bar{c} = \bar{c}_s + \bar{V}_s$, therefore

$$\bar{\bar{P}}_s = \rho_s \langle (\bar{c}_s + \bar{V}_s) (\bar{c}_s + \bar{V}_s) \rangle_s = \rho_s \langle \bar{c}_s \bar{c}_s \rangle_s + \rho_s \langle \bar{V}_s \bar{V}_s \rangle_s$$

and we get $\bar{\bar{P}}_s = \bar{\bar{P}}'_s + \rho_s \langle \bar{V}_s \bar{V}_s \rangle_s$. In many cases the mean velocity of individual species is not that different from the mean fluid velocity, so that the diffusion contribution to the pressure tensor is usually small.

Now, for the remaining parts of the moment equation

$$n_s \langle \bar{w} \cdot \nabla \phi \rangle_s = -\rho_s \langle (\bar{u} + \bar{c}) \cdot \nabla \bar{u} \rangle_s = -\rho_s \bar{u} \cdot \nabla \bar{u} - \rho_s \bar{V}_s \cdot \nabla \bar{u}$$

and the force term is similar to what we obtained in (b)

$$n_s \left\langle \frac{\vec{F}_s}{m_s} \cdot \nabla_w \phi \right\rangle_s = n_s \left\langle \vec{F}_s \cdot \vec{I} \right\rangle_s = n_s q_s (\vec{E} + \vec{u}_s \times \vec{B})$$

Using the definition of the diffusion velocity, we rewrite the term as

$$n_s \left\langle \frac{\vec{F}_s}{m_s} \cdot \nabla_w \phi \right\rangle_s = n_s q_s (\vec{E} + (\vec{u} + \vec{V}_s) \times \vec{B}) = n_s q_s [(\vec{E} + \vec{u} \times \vec{B}) + \vec{V}_s \times \vec{B}] = n_s q_s [\vec{E}' + \vec{V}_s \times \vec{B}]$$

Where \vec{E}' is the electric field as seen in the frame of reference of the fluid moving at a mean velocity \vec{u} . The collision integral in this case is the same as the one found in (b), therefore the complete moment equation is

$$\frac{\partial}{\partial t} (\rho_s \vec{V}_s) + \nabla \cdot (\vec{P}_s + \rho_s \vec{V}_s \vec{u}) + \rho_s \left[\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + \vec{V}_s \cdot \nabla \vec{u} \right] - n_s q_s [\vec{E}' + \vec{V}_s \times \vec{B}] = \sum_r \vec{M}_{rs}$$

Now we add for all species, such that the fluid density, charge and pressure are

$$\sum_s \rho_s = \rho \quad \sum_s n_s q_s = \rho_{ch} \quad \sum_s \vec{P}_s = \vec{P}$$

The summation of collision terms cancels out given de symmetry of momentum transfer $\vec{M}_{rs} = -\vec{M}_{sr}$, while the one over the diffusion velocities is zero by definition

$$\sum_s \sum_r \vec{M}_{rs} = 0 \quad \sum_s \rho_s \vec{V}_s = \sum_s \rho_s (\vec{u}_s - \vec{u}) = \rho \vec{u} - \rho \vec{u} = 0$$

We also define the diffusion current density as $\vec{j}_D = \sum_s n_s q_s \vec{V}_s$. The total current

density would be this plus the contribution of charges moving with the fluid $\vec{j} = \vec{j}_D + \rho_{ch} \vec{u}$. The momentum moment equation for the fluid is finally

$$\rho \left[\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right] + \nabla \cdot \vec{P} = \rho_{ch} \vec{E}' + \vec{j}_D \times \vec{B}$$

or in the absence of net charge, $\rho_{ch} = \sum_s n_s q_s = 0$

$$\rho \left[\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right] + \nabla \cdot \vec{P} = \vec{j} \times \vec{B}$$

e. $\phi = \frac{1}{2} m_s w^2$ (Kinetic energy)

As before, start by noting that $\frac{\partial \phi}{\partial t} = 0$ and $\nabla \phi = 0$. For the first term of the moment equation

$$\frac{\partial}{\partial t} (n_s \langle \phi \rangle_s) = \frac{\partial}{\partial t} \left[\frac{\rho_s}{2} \langle w^2 \rangle_s \right] = \frac{\partial}{\partial t} \left[\frac{\rho_s}{2} \langle (\vec{c}_s + \vec{u}_s) \cdot (\vec{c}_s + \vec{u}_s) \rangle_s \right] = \frac{\partial}{\partial t} \left[\frac{\rho_s}{2} (\langle c_s^2 \rangle_s + u_s^2) \right]$$

and from the definition of temperature $\frac{1}{2} m_s \langle c_s^2 \rangle_s = \frac{3}{2} k T'_s$ we obtain

$$\frac{\partial}{\partial t} (n_s \langle \phi \rangle_s) = \frac{\partial}{\partial t} \left[\frac{\rho_s}{2} u_s^2 + \frac{3}{2} n_s k T'_s \right]$$

Recall that the prime denotes that the quantity (in this case the temperature) is taken with respect to the random velocity of species s . For the next term we have

$$\begin{aligned} \nabla \cdot n_s \langle \phi \vec{w} \rangle_s &= \nabla \cdot \rho_s \left\langle \frac{1}{2} w^2 \vec{w} \right\rangle_s = \nabla \cdot \frac{\rho_s}{2} \langle (c_s^2 + u_s^2 + 2\vec{c}_s \cdot \vec{u}_s) (\vec{c}_s + \vec{u}_s) \rangle_s \\ &= \nabla \cdot \left[\frac{\rho_s}{2} \langle c_s^2 \vec{c}_s \rangle_s + \frac{\rho_s}{2} \langle c_s^2 \rangle_s \vec{u}_s + \frac{\rho_s}{2} u_s^2 \vec{u}_s + \rho_s \langle \vec{c}_s (\vec{c}_s \cdot \vec{u}_s) \rangle_s \right] \end{aligned}$$

Defining the heat flux (also with respect to the random velocity of species s)

$$\vec{q}'_s = \frac{\rho_s}{2} \langle c_s^2 \vec{c}_s \rangle_s$$

and noting that (using index notation)

$$\rho_s \langle \vec{c}_s (\vec{c}_s \cdot \vec{u}_s) \rangle_s = \rho_s \langle c_i c_j u_j \rangle_s = \rho_s \langle c_i c_j \rangle_s u_j = P'_{ij} u_j = \vec{P}'_s \vec{u}_s$$

Therefore we have

$$\nabla \cdot n_s \langle \phi \vec{w} \rangle_s = \nabla \cdot \left[\vec{q}'_s + n_s \vec{u}_s \frac{3}{2} k T'_s + \frac{\rho_s}{2} u_s^2 \vec{u}_s + \vec{P}'_s \vec{u}_s \right]$$

For the force term of the moment equation, we have

$$n_s \left\langle \frac{\vec{F}_s}{m_s} \cdot \nabla_w \phi \right\rangle_s = n_s \langle \vec{F}_s \cdot \vec{w} \rangle_s = n_s q_s \langle (\vec{E} + \vec{w} \times \vec{B}) \cdot \vec{w} \rangle_s = n_s q_s \langle \vec{E} \cdot \vec{w} \rangle_s = n_s q_s \vec{E} \cdot \vec{u}_s = \vec{E} \cdot \vec{j}_s$$

Where \vec{j}_s is the mean current carried by species s . Finally, for the collision term we observe that (keep in mind that the magnitude of the relative velocity vector does not change)

$$\begin{aligned} \phi' - \phi &= \frac{1}{2} m_s [w'^2 - w^2] = \frac{1}{2} m_s \left[\left(\vec{G} - \frac{m_r}{m_r + m_s} \vec{g}' \right)^2 - \left(\vec{G} - \frac{m_r}{m_r + m_s} \vec{g} \right)^2 \right] \\ &= \frac{m_s m_r}{m_r + m_s} \vec{G} \cdot (\vec{g} - \vec{g}') = \mu_{sr} \vec{G} \cdot (\vec{g} - \vec{g}') \end{aligned}$$

We write the collision integral as

$$E_{rs} = \mu_{rs} \int_{w_1} \int_w f_s f_{r_1} g d^3 w d^3 w_1 \int_{\Omega} \sigma_{rs} \vec{G} \cdot (\vec{g} - \vec{g}') d\Omega$$

Following the same reasoning as in case (b), we obtain for the momentum transfer cross section

$$Q_{rs}^{(1)}(g) = \int_{\Omega} \sigma_{rs}(\chi, g) (1 - \cos \chi) d\Omega$$

So that the collision integral reduces to

$$E_{rs} = \mu_{rs} \int_{w_1} \int_w f_s f_{r_1} g \vec{G} \cdot \vec{g} Q_{rs}^{(1)}(g) d^3 w d^3 w_1$$

Putting all the terms together we find the Eulerian form of the kinetic energy moment equation

$$\frac{\partial}{\partial t} \left[\frac{\rho_s}{2} u_s^2 + \frac{3}{2} n_s k T'_s \right] + \nabla \cdot \left[\vec{q}'_s + n_s \vec{u}_s \frac{3}{2} k T'_s + \frac{\rho_s}{2} u_s^2 \vec{u}_s + \vec{P}'_s \vec{u}_s \right] - \vec{E} \cdot \vec{j}_s = \sum_r E_{rs}$$

Rearranging to put it in a more interesting way

$$\frac{\partial}{\partial t} \left[n_s \left(\frac{1}{2} m_s u_s^2 + \frac{3}{2} k T'_s \right) \right] + \nabla \cdot \left[n_s \vec{u}_s \left(\frac{1}{2} m_s u_s^2 + \frac{3}{2} k T'_s \right) \right] + \nabla \cdot \left[\vec{q}'_s + \vec{P}'_s \vec{u}_s \right] = \vec{E} \cdot \vec{j}_s + \sum_r E_{rs}$$

In this way, the two terms in the RHS can be considered as “inputs” to the energy equation described in the LHS.

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