16.410/413
Principles of Autonomy and Decision Making

Lecture 24: Sequential Games

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Outline

1. Game Theory
   - Overview
   - Games in normal form: Nash equilibria, pure and mixed strategies
   - Games in extensive form
2. Sequential Games

Game Theory

Games

- Multiple "players" independently choose actions, based on the available information, to pursue individual goals.
- Created by John Von Neumann in the late 1920s.

Applications

- Economics
- Political Science/Diplomacy/Military Strategy
- Biology
- Computer Science/Artificial Intelligence
- Computer games
- Resource allocation in networks (internet, cell phones,...)
- Robust control (disturbance rejection)
- Air traffic collision avoidance
- UAV Pursuit-evasion

Types of Games

Zero-sum games

All the gains/losses of a player are exactly balanced by the gains/losses of all other players (possibly modulo a constant).

- Zero-sum: a game of chess, tic-tac-toe, rock/paper/scissors, poker (with no house cut), risk, dividing a cake, presidential election, dogfights (?).
- Non-zero sum: contract negotiation, trade agreements, chicken and hawk/dove game, prisoners dilemma, MMORPGs, dogfights (?).

Cooperative vs. non-Cooperative Games

- A game is cooperative if groups of players may enforce binding agreements. (E.g., through a third party, such as a legal system.)
- A game is non-cooperative if no such binding agreements exist. Cooperation may occur, but is self-serving.

Types of Games, cont'd.

Symmetric games

The game is invariant to relabeling on the players.

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In a sequential game, the players act at well-defined turns, and have some information on what the other(s) did at previous turns. In a simultaneous game, all players act at the same time, or equivalently, have no information on the actions of the others in the same turn.

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Are the games listed above symmetric/sequential/perfect information games?

Games in normal form

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Suitable for simultaneous games, or for summarizing the effects of "strategies."

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Prisoner's dilemma

Two suspects ("players") are arrested and accused of a crime. Since the police do not have enough evidence, they can be convicted only if at least one of the suspects testifies against the other.

	Player B cooperates	**Player B defects**
Player A cooperates	(-1,-1)	(-10,0)
Player A defects	(0,-10)	(-5,-5)

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Nash equilibria

- A Nash equilibrium is a choice of strategies such that no player can gain by unilaterally changing his/her strategy.
- Nash equilibria are not necessarily efficient.

Pure and Mixed strategies

Rock-Paper-Scissors

Rock beats Scissors beats Paper beats Rock.

A/B	Rock	Scissors	Paper
Rock	(0,0)	(1,-1)	(-1,1)
Scissors	(-1,1)	(0,0)	(1,-1)
Paper	(1,-1)	(-1,1)	(0,0)

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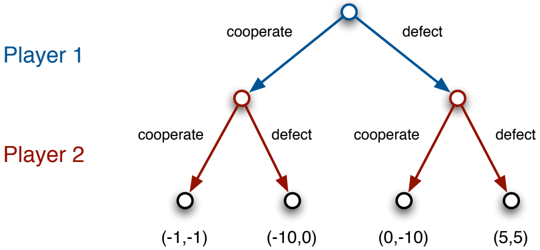
Repeated games and randomized strategies

- Nash proved that any finite game has at least a Nash equilibrium. However, such a Nash equilibrium is not necessarily **pure**, i.e., deterministically defined (each player adopts one strategy).
- In a **mixed** strategy, a player chooses his/her strategy randomly according to a given probability distribution.
- Rock-Paper-Scissors is a typical example of a game with a mixed Nash equilibrium.

Games in extensive form

Extensive Form

- Suitable for games played in sequential "turns."
- Consider the following version of the prisoner's dilemma: is it the same as the one we saw before?



Outline

Zero-Sum Two-Player Sequential Games

Key characteristics

- Two players;
- Zero-sum reward structure (the reward of a player is the cost for the other).
- Sequential moves (from a finite set);
- Perfect information;
- The game terminates in a finite number of steps, no matter how it is played.

Problem data

- An initial state (incl. whose turn it is);
- One or more terminal states;
- State/action pairs;
- The cost/reward associated with terminal states.

Objective

Compute, for each player, a strategy that associates to each state an action that maximizes the reward if the other player plays rationally.

Tic-Tac-Toe

Initial state

Empty board, **X** to go first.

Actions

Place **X** (or **O**) in an empty square.

Terminal states

- Three **X**s or **O**s on the same line is a win.
- No empty squares is a tie.

Reward (at terminal state)

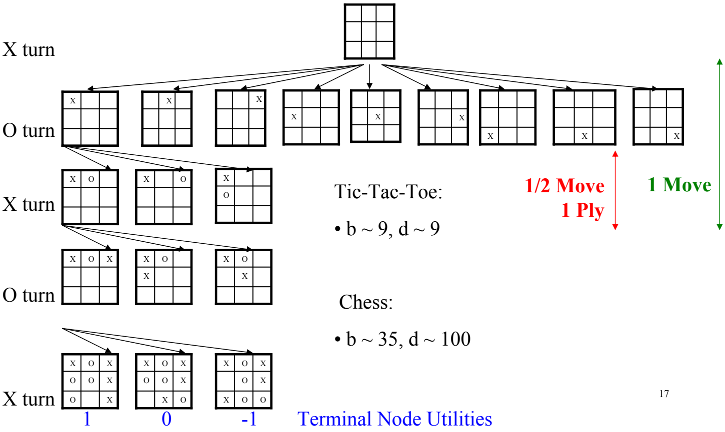
- 1 for a win, 0 for a tie, -1 for a loss.

X	X	O
	O	
X		

Notes

- Max depth: 9 "plies" (i.e., moves)
- Branching: at most $(10 - i)$ possible moves at the i-th ply.

Tic-Tac-Toe Game Tree



The complete tree has no more than 9! $= 362880$ nodes
(not accounting for symmetries and termination conditions).

Tree search for a single player

Tic-Tac-Toe

- Let us assume that we can construct the whole tree representing all possible play sequences.
- If you were playing "solitaire tic-tac-toe," you would choose one of the branches leading to a win (max reward), and place Xs, and Os consequently.
- Unfortunately, you do not get to choose the Os!
- Your adversary, if he/she had to choose, would seek to minimize your reward (i.e., maximize his/her own)!

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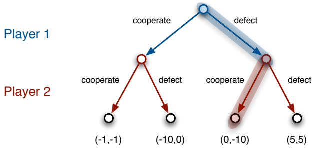
Sequential prisoner's dilemma (note: not zero-sum)

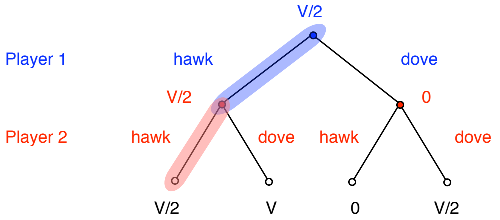
- In the sequential prisoner dilemma game, the best plan for the first player is to defect and make the other player cooperate.
- The other player may not agree...
- in fact, it will be better for him/her to defect!



Two-player search: Min-Max

- Each player tries to **maximize his/her own reward**, **assuming that the other player uses an optimal strategy** (for his/her own reward)
- In a zero-sum game, this is equivalent to saying that Player 1 is trying to maximize his/her reward, and Player 2 is trying to minimize Player 1's reward
  i.e., Player 1 MAXimizes, Player 2 MINimizes.
- In practice:
  - build the whole tree, find terminal states and evaluate the corresponding rewards;
  - Moving backwards from the leaves, associate to parent nodes the MIN or MAX value of all their children (depending on whose turn it is).

Hawk/Dove game with no cost for confrontation



Practical Considerations

The MinMax (or MiniMax) algorithm finds optimal strategies. However, it requires building/searching the complete game tree.

- Tic-Tac-Toe: about 10^5 nodes.
- Chess: about $35^{100} = 2.5 \times 10^{154}$ nodes!

In order to limit the complexity of the search, build a partial tree, i.e., a tree whose leaves are not necessarily terminal states. Two problems:

- How do we choose when to stop expanding the tree?
  - Fixed cut-off, e.g., depth d.
  - Iterative deepening.
- What value do we associate to the leaves?
  - Use "evaluation functions," ideally designed to give a good estimate of the terminal reward given an intermediate state (same function as heuristic functions).
  - E.g., in chess, you can give a numeric value to each piece in play.

Alpha-Beta Pruning

- Still, the complexity of a good search might be excessive.
- Performance gains can be attained by using **branch and bound** techniques, removing from the search subtrees that are guaranteed to be no better than others already discovered (w.r.t. the actual reward, or the evaluation function, depending on the tree construction).
- In practice:
  - Associate to each node an interval in which the reward can lie. Initialize with $(-\infty, +\infty)$.
  - Do a depth first search, tightening the bounds for the reward, i.e., $[\alpha, \beta]$.
  - If a node provably cannot offer any improvements, prune (i.e., do not search further) the corresponding subtree.

Characteristics of Alpha-Beta Pruning

What are the α and β values of a vertex *s*?

- α: this represent the largest known lower bound on the value of the game if it started at the vertex *s* (the value of *s*).
- β: this represent the smallest known upper bound on the value of *s*.

Initial values of (α, β) for a vertex *s*

- If *s* is the root of the tree: $(\alpha, \beta) = (-\infty, \infty)$.
- If *s* is a terminal state, i.e., a leaf of the tree: $\alpha = \beta =$ value of *s*.

Properties of (α, β) for a vertex *s*

- α never decreases.
- β never increases.

Pseudocode for alpha-beta

minimax(node, player, depth)

```
return alphabeta(node, player, depth, −∞, ∞)
```

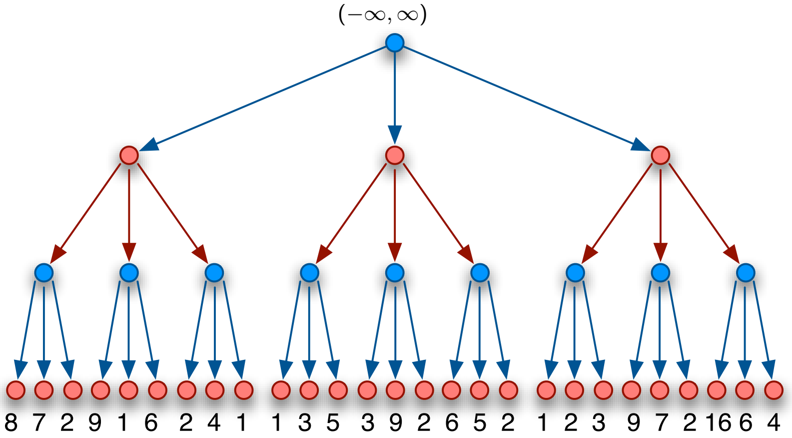
alphabeta(node, player, depth, α, β)

```
if node is a terminal node, or depth = 0 then
    return the (heuristic) value of the node
foreach child of node do
    if player == MAX then
        aux = alphabeta(child, MIN, depth-1, α, β));
        if  aux > α then α = aux ;                          // Adjust the bound
        if α > β then break;                             // No reason to continue...
        return α ;                    // This is the best result for MAX from here
    else
        aux = alphabeta(child, MAX, depth-1, α, β));
        if  aux < β then β = aux ;                          // Adjust the bound
        if α > β then break;                             // No reason to continue...
        return β ;                    // This is the best result for MIN from here
```

Alpha-Beta in practice

- Visit the vertices of the tree in Depth-First Search order.
- At the first visit of a MAX node, set its β value to the β value of its parent.
- At the first visit of a MIN node, set its α value to the α value of its parent.
- Every time a MAX node is revisited, update its α value to the maximum known value of its children.
- Every time a MIN node is revisited, update its β value to the minimum known value of its children.
- If at any point it happens that $\alpha \geq \beta$, it means that that particular vertex cannot be part of an optimal solution, since there is at least another solution that is certainly no worse than any solution containing the vertex $\Rightarrow$ there is no point in further investigating the subtree rooted at that vertex $\Rightarrow$ **PRUNE THE SUBTREE**.
- When leaving a vertex *s* for the last time (i.e., when moving back towards the root), set its value to α if *s* is a MAX node, or to β if *s* is a MIN node.

Alpha-Beta Pruning Example



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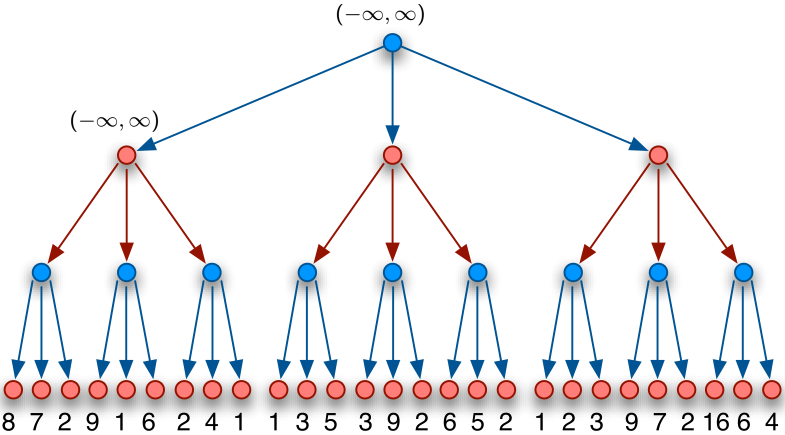

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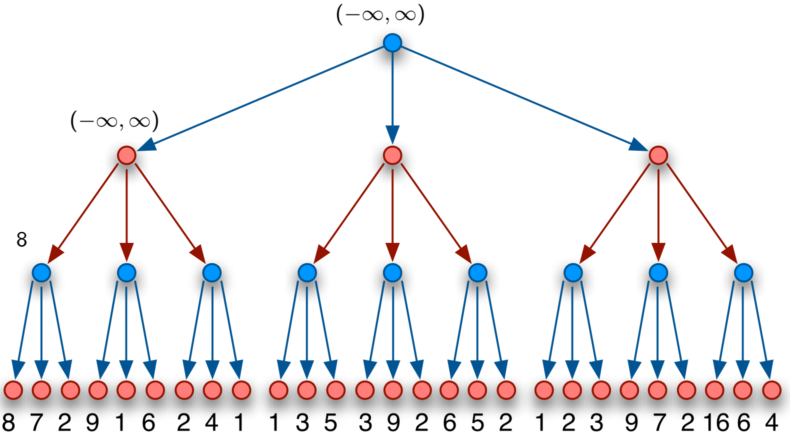

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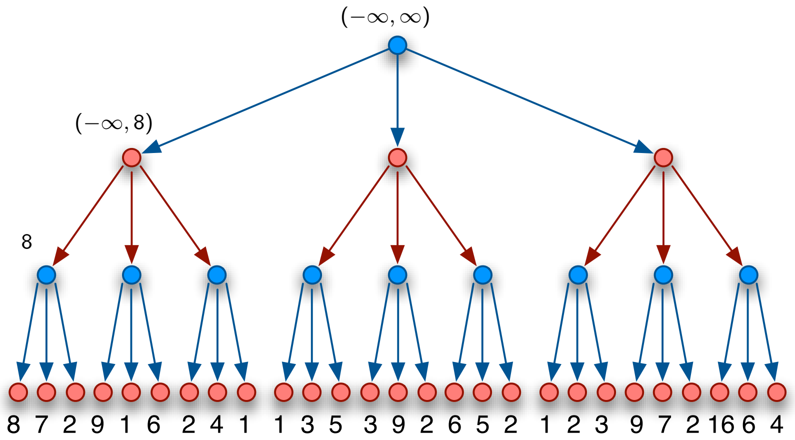

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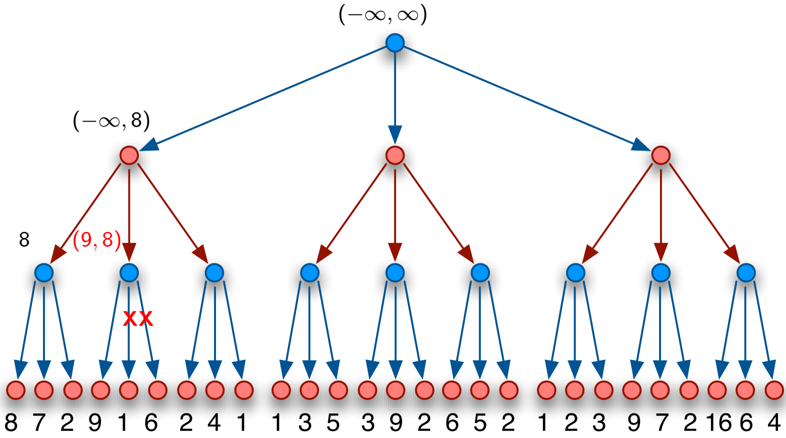

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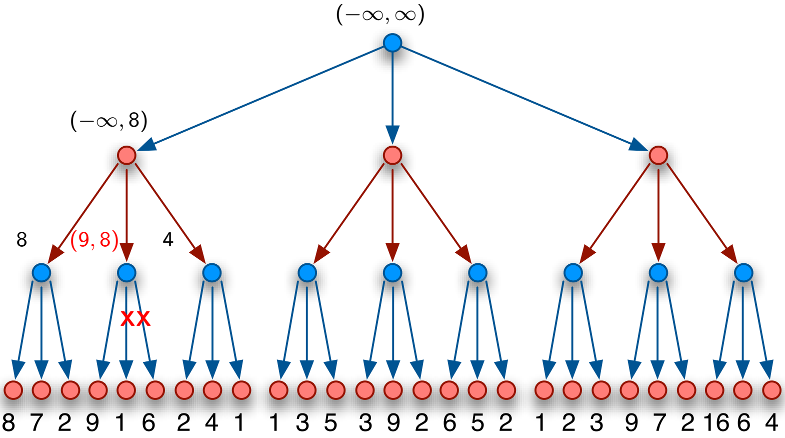

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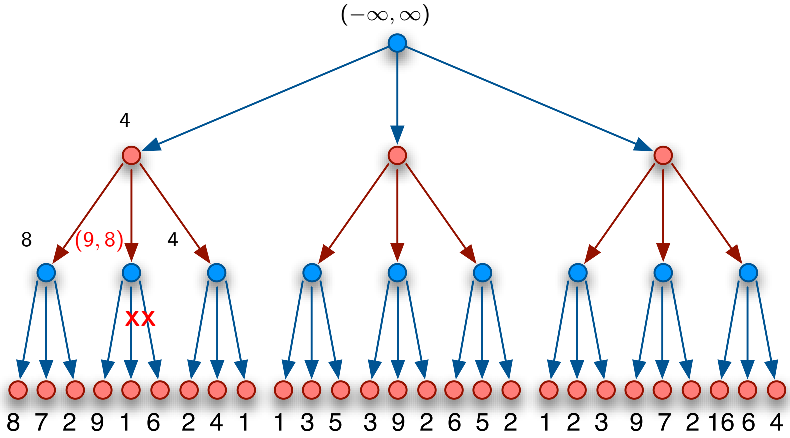

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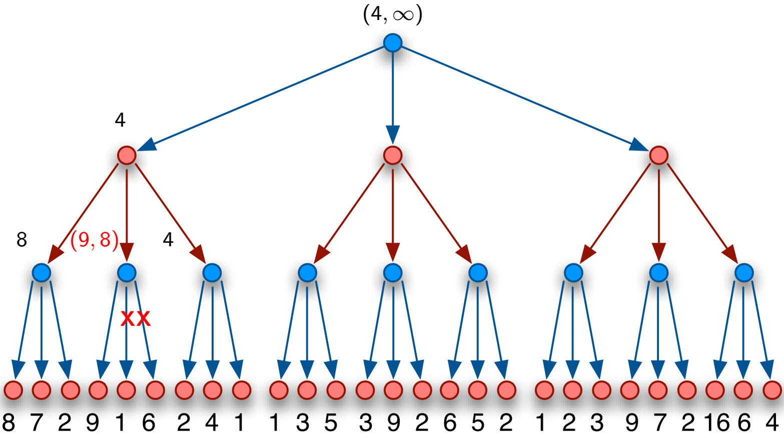

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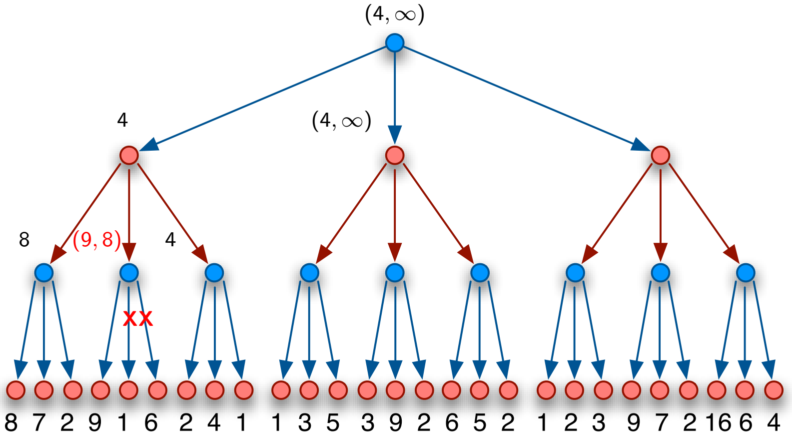

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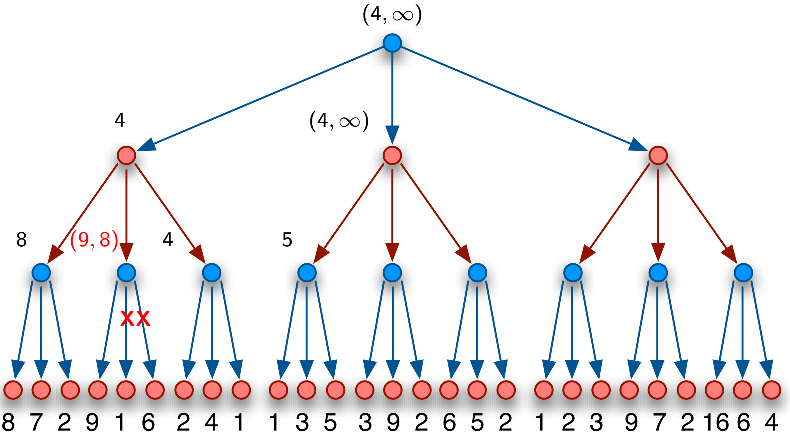

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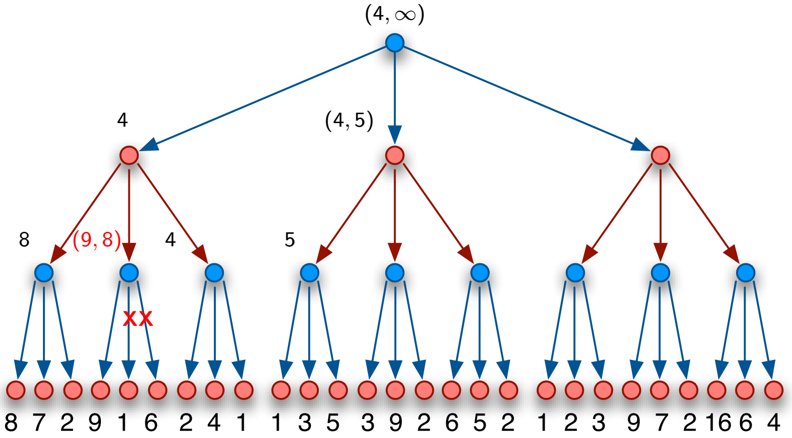

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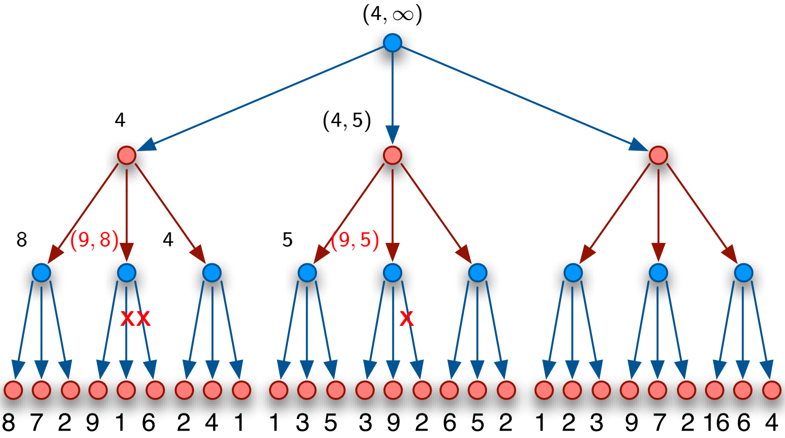

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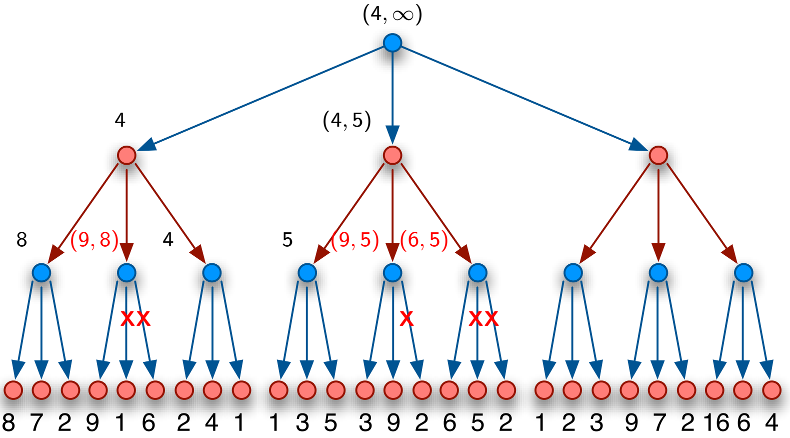

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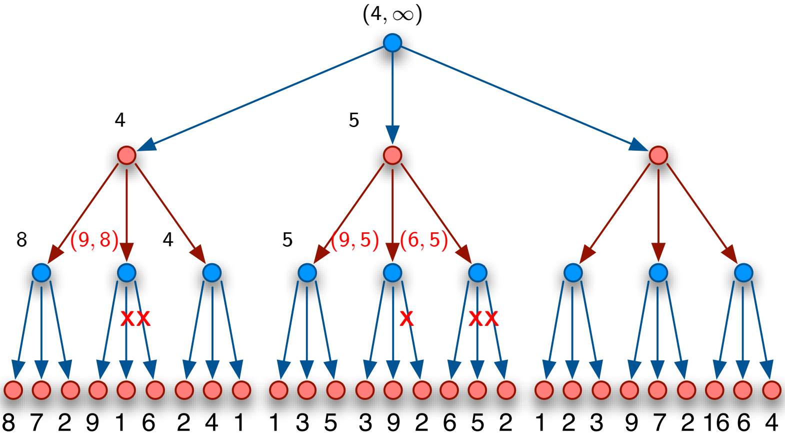

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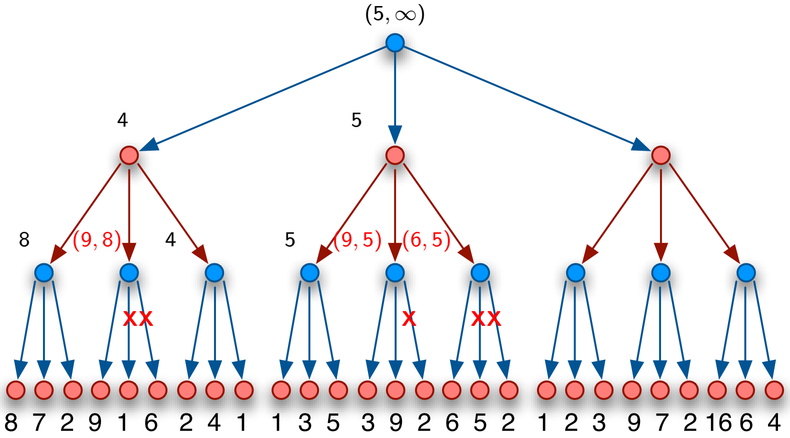

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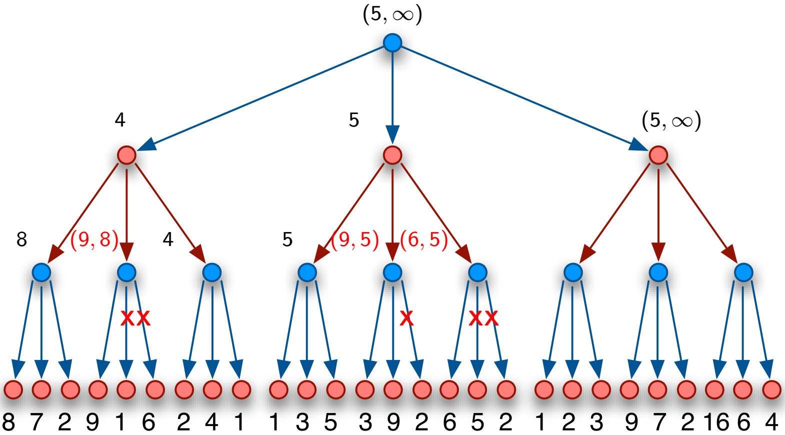

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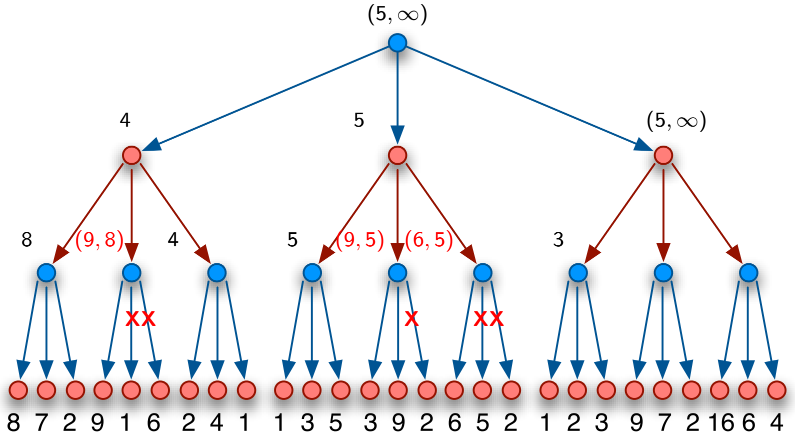

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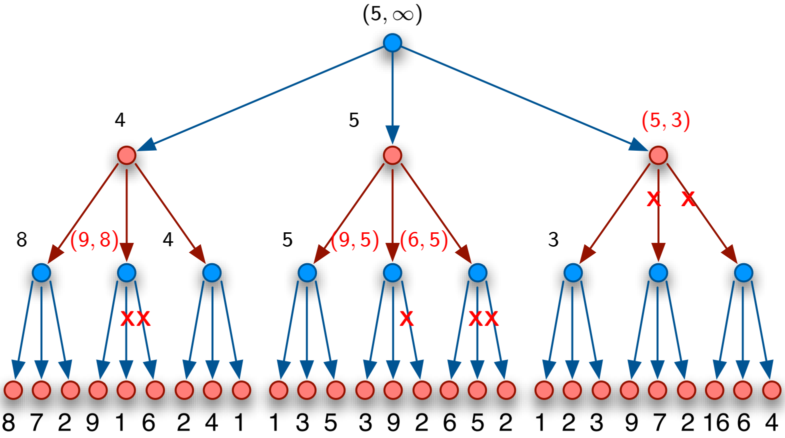

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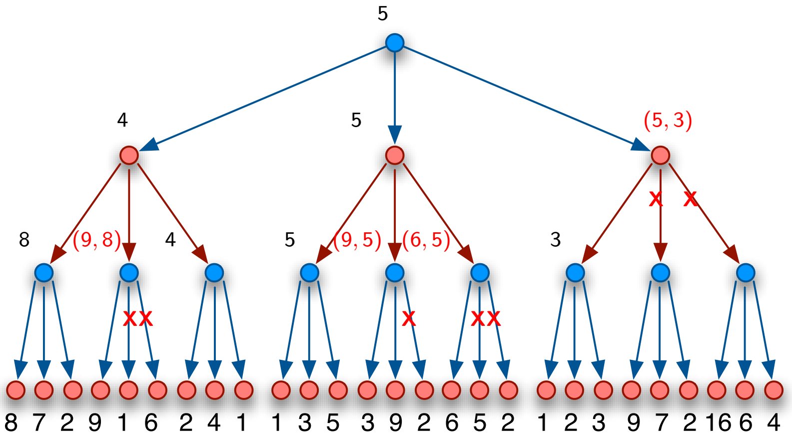

Effectiveness of Alpha-Beta Pruning

- The performance of Alpha-Beta pruning depends strongly on the order in which the tree is searched.
- Ideally, one would want to examine the best successors first.
(Clearly, this is not achievable, since if we knew the best successors a priori, we would have solved the problem!)
- If this can be done, alpha-beta searches only need $O(b^{d/2})$ time
(compare with standard minmax, a depth-first search requiring $O(b^d)$ time).
- Effectively, this allows to double the search depth!
- State-of-the-art algorithms, such as NegaScout and MTD(f) are based on alpha-beta pruning, combined with null-window searches, which can yield quickly bounds on the value of the game.