16.346 Astrodynamics
Fall 2008

Clohessy-Wiltshire Equations†

We begin with the equations for the restricted three-body problem

$$m\left[\frac{d^2\mathbf{r}}{dt^2} + 2\boldsymbol{\omega}\times\frac{d\mathbf{r}}{dt} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})\right] = -\frac{Gmm_1}{\rho_1^3}\boldsymbol{\rho}_1 - \frac{Gmm_2}{\rho_2^3}\boldsymbol{\rho}_2$$

where

$$\boldsymbol{\rho}_1 = \mathbf{r}-\mathbf{r}_1 \qquad \boldsymbol{\rho}_2 = \mathbf{r}-\mathbf{r}_2 \qquad \mathbf{r} = \mathbf{r}_3 - \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1+m_2}$$

$$\boldsymbol{\omega} = \omega\,\mathbf{i}_\zeta \quad \text{with} \quad \omega^2 = \frac{G(m_1+m_2+m)}{r_{12}^3} \approx \frac{G(m_1+m_2)}{r_{12}^3}$$

With m_1 and m_2 on ξ-axis, then $\mathbf{r}_1 = r_1\,\mathbf{i}_\xi$ and $\mathbf{r}_2 = r_2\,\mathbf{i}_\xi$

To adapt these equations to the problem of a chase spacecraft m in pursuit of a target spacecraft m_1 both moving about a central body of mass m_2, let both m and m_1 become infinitesimal. As a result r_2 will be zero so that $\mathbf{r}$ and $\boldsymbol{\rho}_2$ are the same vector. The vector $\boldsymbol{\rho}_1 \equiv \boldsymbol{\rho}$ is the position of the chase spacecraft relative to the target spacecraft. Further, the angular velocity is

$$\omega^2 = \frac{Gm_2}{r_1^3} \qquad \text{or} \qquad \omega^2 r_1^3 = Gm_2$$

so that the equations of motion of the chase spacecraft can be written as

$$\boxed{\frac{d^2\boldsymbol{\rho}}{dt^2} + 2\boldsymbol{\omega}\times\frac{d\boldsymbol{\rho}}{dt} + \boldsymbol{\omega}\times[\boldsymbol{\omega}\times(\boldsymbol{\rho}+\mathbf{r}_1)] = -\frac{\omega^2 r_1^3}{r^3}\mathbf{r}}$$

where $\boldsymbol{\rho}$ and $\mathbf{r}_1 = r_1\,\mathbf{i}_\xi$ are the position vectors of the chase and target spacecrafts, respectively.

Note: $\mathbf{r} = \boldsymbol{\rho} + \mathbf{r}_1$.

This differential equation is non-linear because of the factor $1/r^3$. However, with the use of the Taylor Series expansion, we write

$$\frac{r^2}{r_1^2} = \frac{(\boldsymbol{\rho}+\mathbf{r}_1)\cdot(\boldsymbol{\rho}+\mathbf{r}_1)}{r_1^2} = \frac{\rho^2 + 2\boldsymbol{\rho}\cdot\mathbf{r}_1 + r_1^2}{r_1^2} = 1 + 2x\,\mathbf{i}_\xi\cdot\mathbf{i}_{r_1} + x^2$$

$$\frac{r_1}{r} = (1 + 2\,\mathbf{i}_\xi\cdot\mathbf{i}_{r_1}x + x^2)^{-\frac{1}{2}} = 1 - \mathbf{i}_\xi\cdot\mathbf{i}_{r_1}x + \cdots$$

where $x = \dfrac{\rho}{r_1}$. Therefore,

$$\frac{r_1^3}{r^3} = 1 - 3\,\mathbf{i}_\xi\cdot\mathbf{i}_\rho\frac{\rho}{r_1} + O\left(\frac{\rho^2}{r_1^2}\right)$$

† W.H. Clohessy and R.S. Wiltshire, Journal of Aerospace Sciences, Vol. 27, No. 9, 1960, pp. 653–658.

and the equation will be linear if we ignore the higher order terms. Then

$$\frac{d^2\boldsymbol{\rho}}{dt^2} + 2\boldsymbol{\omega} \times \frac{d\boldsymbol{\rho}}{dt} + \boldsymbol{\omega} \times [\boldsymbol{\omega} \times (\boldsymbol{\rho} + \mathbf{r}_1)] = -\omega^2 \left[1 - 3(\mathbf{i}_\xi \cdot \boldsymbol{\rho})\frac{1}{r_1}\right](\boldsymbol{\rho} + \mathbf{r}_1)$$

reduces to

$$\frac{d^2\boldsymbol{\rho}}{dt^2} + 2\boldsymbol{\omega} \times \frac{d\boldsymbol{\rho}}{dt} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) = -\omega^2 \boldsymbol{\rho} + 3\omega^2(\mathbf{i}_\xi \cdot \boldsymbol{\rho})\mathbf{i}_\xi + O(\rho^2)$$

since the term with the factor $(\mathbf{i}_\xi \cdot \boldsymbol{\rho})\boldsymbol{\rho}$ is $O(\rho^2)$.

Finally,

$$\boldsymbol{\rho} = \xi\, i_\xi + \eta\, \mathbf{i}_\eta + \zeta\, i_\zeta$$

so that

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) = -\omega^2(\xi\, i_\xi + \eta\, \mathbf{i}_\eta) \qquad \text{and} \qquad \mathbf{i}_\xi \cdot \boldsymbol{\rho} = \xi$$

Therefore, the differential equation for **the motion of the chase spacecraft relative to the target spacecraft** is

$$\boxed{\frac{d^2\boldsymbol{\rho}}{dt^2} + 2\boldsymbol{\omega} \times \frac{d\boldsymbol{\rho}}{dt} = -\omega^2\zeta\, \mathbf{i}_\zeta + 3\omega^2\xi\, \mathbf{i}_\xi + O(\rho^2)}$$

or in scalar form

$$\boxed{\begin{aligned} \frac{d^2\xi}{dt^2} - 2\omega\frac{d\eta}{dt} - 3\omega^2\xi &= 0 \\ \frac{d^2\eta}{dt^2} + 2\omega\frac{d\xi}{dt} &= 0 \\ \frac{d^2\zeta}{dt^2} + \omega^2\zeta &= 0 \end{aligned}}$$

It is sometimes convenient to express the position vector

$$\boldsymbol{\rho} \equiv \mathbf{r} = x\,\mathbf{i}_\theta + y\,\mathbf{i}_r - z\,\mathbf{i}_z \qquad \mathbf{i}_{r_1} = \mathbf{i}_r \qquad \boldsymbol{\omega} = -\omega\,\mathbf{i}_z$$

with x in the direction of motion $\mathbf{i}_\theta$, y in the radial direction $\mathbf{i}_r$ and $\mathbf{i}_z = \mathbf{i}_\theta \times \mathbf{i}_r$ normal to the orbital plane. Then the equations of motion are‡ are

$$\boxed{\begin{aligned} \frac{d^2x}{dt^2} + 2\omega\frac{dy}{dt} &= 0 \\ \frac{d^2y}{dt^2} - 2\omega\frac{dx}{dt} - 3\omega^2 y &= 0 \\ \frac{d^2z}{dt^2} + \omega^2 z &= 0 \end{aligned}}$$

The Clohessy-Wiltshire equations are three simultaneous second-order, linear, constant-coefficient, coupled differential equations which are capable of exact solution.

‡ S.W. Shepperd, Journal of Guidance, Control, and Dynamics, Vol. 14, No. 6, 1991, pp. 1318–1322.

General Solution of the C-W Equations

Introduce the dimensionless time variable $\tau = \omega t$ so that the Clohessy-Wiltshire equations take the form

$$\frac{d^2x}{d\tau^2} + 2\frac{dy}{d\tau} = 0$$

$$\frac{d^2y}{d\tau^2} - 2\frac{dx}{d\tau} - 3y = 0$$

$$\frac{d^2z}{d\tau^2} + z = 0$$

The general solution of these equations, with initial conditions x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$ and $\dot{z}_0$ and using the notation $\dfrac{dx}{d\tau} = \dot{x}$, $\dfrac{dy}{d\tau} = \dot{y}$ and $\dfrac{dz}{d\tau} = \dot{z}$, is

$$\begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 6\sin\tau - 6\tau & 4\sin\tau - 3\tau & 2\cos\tau - 2 \\ 0 & 4 - 3\cos\tau & 2 - 2\cos\tau & \sin\tau \\ 0 & 6\cos\tau - 6 & 4\cos\tau - 3 & -2\sin\tau \\ 0 & 3\sin\tau & 2\sin\tau & \cos\tau \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ \dot{x}_0 \\ \dot{y}_0 \end{bmatrix}$$

$$\begin{bmatrix} z \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{bmatrix} \begin{bmatrix} z_0 \\ \dot{z}_0 \end{bmatrix}$$