16.346 Astrodynamics
Fall 2008

Developing the Algorithm

The right hand side of the cubic equation

$$y^3 - y^2 = m\frac{E - \sin E}{4\tan^3 \frac{1}{2}E}$$

can be expressed as the derivative of a hypergeometric function

Utilizing series manipulations, we find

$$\tfrac{1}{2}\sin E = \frac{\tan\frac{1}{2}E}{1+x} = \tan\tfrac{1}{2}E(1 - x + x^2 - \cdots)$$

$$\tfrac{1}{2}E = \tan\tfrac{1}{2}E(1 - \tfrac{1}{3}x + \tfrac{1}{5}x^2 - \cdots)$$

$$\frac{E - \sin E}{4\tan^3\frac{1}{2}E} = \tfrac{1}{3} - \tfrac{2}{5}x + \tfrac{3}{7}x^2 - \tfrac{4}{9}x^3 + \cdots = -\frac{d}{dx}F(\tfrac{1}{2}, 1; \tfrac{3}{2}; -x)$$

where the hypergeometric function and its derivative are

$$F(\tfrac{1}{2}, 1; \tfrac{3}{2}; -x) = \frac{\arctan\sqrt{x}}{\sqrt{x}} = (1 - \tfrac{1}{3}x + \tfrac{1}{5}x^2 - \tfrac{1}{7}x^3 + \tfrac{1}{9}x^4 - \cdots)$$

$$\frac{dF}{dx} = \frac{d}{dx}\left(\frac{\arctan\sqrt{x}}{\sqrt{x}}\right) = \frac{1}{2x}\left(\frac{\arctan\sqrt{x}}{\sqrt{x}} - \frac{1}{1+x}\right) = -(\tfrac{1}{3} - \tfrac{2}{5}x + \tfrac{3}{7}x^2 - \tfrac{4}{9}x^3 + \cdots)$$

The function $F(\frac{1}{2}, 1; \frac{3}{2}; -x)$ also has a continued fraction representation

$$F(\tfrac{1}{2}, 1; \tfrac{3}{2}; -x) = \frac{\arctan\sqrt{x}}{\sqrt{x}} = \cfrac{1}{1 + \cfrac{x}{3 + \cfrac{4x}{5 + \cfrac{9x}{7 + \ddots}}}} = \frac{1}{1 + xG}$$

$$G = \cfrac{1}{3 + \cfrac{4x}{5 + \cfrac{9x}{7 + \cfrac{16x}{9 + \ddots}}}} = \cfrac{1}{3 + \cfrac{4x}{\xi}} \quad \text{where} \quad \xi = 5 + \cfrac{9x}{7 + \cfrac{16x}{9 + \cfrac{25x}{11 + \cfrac{36x}{13 + \ddots}}}}$$

$$\frac{dF}{dx} = \frac{1}{2x}\left(F - \frac{1}{1+x}\right) = -\frac{(1-G)F}{2(1+x)} = -\frac{(2x+\xi)FG}{\xi(1+x)} = -\frac{2x+\xi}{(1+x)[4x+\xi(3+x)]}$$

Finally, a successive substitution algorithm, paralleling the Gauss algorithm, results using:

$$\boxed{x = \sqrt{\left(\frac{1-\ell}{2}\right)^2 + \frac{m}{y^2}} - \frac{1+\ell}{2}} \quad \text{and} \quad \boxed{y^3 - y^2 = \frac{m(2x+\xi)}{(1+x)[4x+\xi(3+x)]}}$$

Graphics of a Successive Substitution Algorithm

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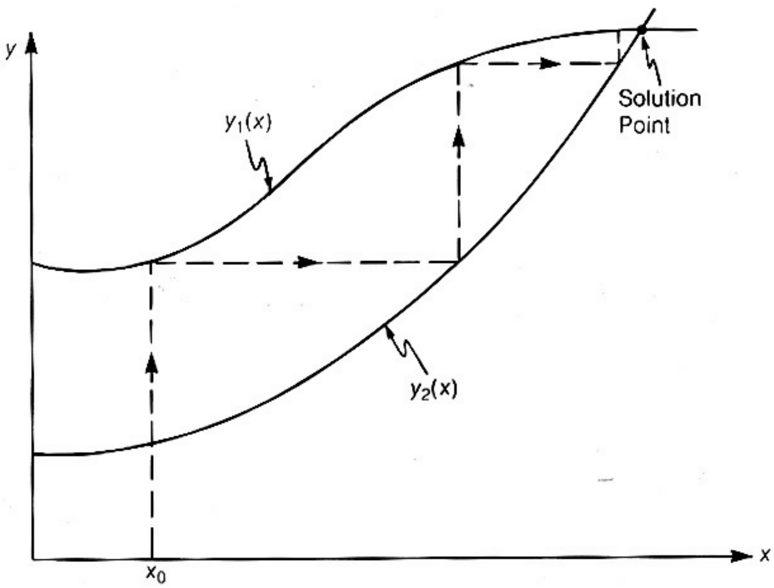

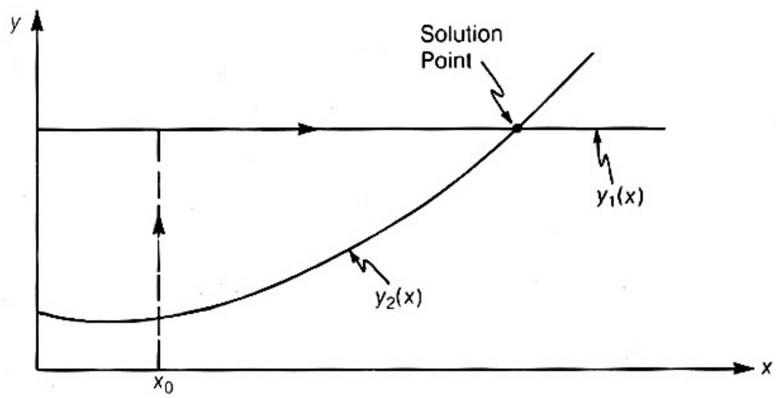

Fig. 7.5 from *An Introduction to the Mathematics and Methods of Astrodynamics.* Courtesy of AIAA. Used with permission.

Improving the Convergence of the New Algorithm

1. Time equation with free parameter β

$$\frac{1}{2}\sqrt{\frac{\mu}{a^3}}(t_2 - t_1) = \left(1 + \beta\,\frac{r_0}{a}\right)(\psi - \sin\psi) + \frac{r_0}{a}\left[\sin\psi - \beta(\psi - \sin\psi)\right]$$

2. With $\qquad F = \dfrac{\arctan\sqrt{x}}{\sqrt{x}} \qquad$ and $\qquad h_1 = 2\beta x(1+x)\dfrac{dF}{dx}$

the time equation can be written as

$$y^3 - y^2 - h_1(x)y^2 - h_2(x) = 0 \qquad \text{where} \qquad h_2(x) = -m\left[\frac{dF}{dx} + \frac{h_1(x)}{(\ell + x)(1+x)}\right]$$

3. Differentiate and obtain

$$\left[3y^2 - 2(1+h_1)y\right]\frac{dy}{dx} - \left[y^2 - \frac{m}{(\ell+x)(1+x)}\right]\frac{dh_1}{dx} + m\left[\frac{d^2F}{dx^2} + h_1(x)\frac{d}{dx}\frac{1}{(\ell+x)(1+x)}\right] = 0$$

At the solution point, the coefficient of dh_1/dx is, of course, zero.
Then, for $dy/dx = 0$ at solution point, we must have

$$\frac{d^2F}{dx^2} + h_1(x)\frac{d}{dx}\frac{1}{(\ell+x)(1+x)} = 0$$

from which we calculate $h_1(x)$ [or $\beta(x)$ which we don't really need] and finally $h_2(x)$.

Battin–Vaughan Algorithm

Given r_1, r_2, θ, $\sqrt{\mu}\,(t_2 - t_1) \equiv k(t_2 - t_1)$

1. Calculate

$$A = \tfrac{1}{2}(r_1 + r_2) \qquad B = \sqrt{r_1 r_2}\cos\tfrac{1}{2}\theta \qquad C = A + B \qquad \ell = \frac{A - B}{C} \qquad m = \frac{[k(t_2 - t_1)]^2}{C^3}$$

2. Initialize $x = \begin{cases} 0 & \text{parabola, hyperbola} \\ \ell & \text{ellipse} \end{cases}$

3. Calculate

$$\xi(x) = 5 + \cfrac{9x}{7 + \cfrac{16x}{9 + \cfrac{25x}{11 + \cfrac{36x}{13 + \ddots}}}}$$

Note: Instead of the continued fraction for $\xi(x)$, we can use

$$\xi(x) = \frac{4x(\sqrt{x} - \arctan\sqrt{x}\,)}{(3 + x)\arctan\sqrt{x} - 3\sqrt{x}}$$

for elliptic orbits which are not close to parabolic.

4. Calculate $H = (1 + 2x + \ell)[4x + (3 + x)\xi(x)]$

$$h_1(x) = \frac{1}{H}\,(\ell + x)^2[1 + 3x + \xi(x)] \qquad \text{and} \qquad h_2(x) = \frac{m}{H}\,[x - \ell + \xi(x)]$$

5. Solve the cubic $y^3 - y^2 - h_1 y^2 - h_2 = 0$

Note:

$$y = \frac{2}{3}(1 + h_1)\left(\frac{b}{z} + 1\right) \qquad \text{converts the cubic to} \qquad z^3 - 3z = 2b$$

$$\text{where} \qquad b = \sqrt{\frac{27h_2}{4(1 + h_1)^3} + 1} \qquad \text{and} \qquad z = \begin{cases} 2\cosh(\frac{1}{3}\operatorname{arccosh} b) & b \ge 1 \\ 2\cos(\frac{1}{3}\arccos b) & b < 1 \end{cases}$$

6. Determine new $x = \sqrt{\dfrac{B^2}{C^2} + \dfrac{m}{y^2}} - \dfrac{A}{C}$

7. Repeat until x no longer changes.

8. Calculate the orbital elements:

$$\frac{1}{a} = \frac{4xy^2}{Cm} \qquad \frac{p}{p_m} = \frac{cy^2(1 + x)^2}{2Cm} \qquad \text{or} \qquad p = \frac{r_1 r_2 y^2 (1 + x)^2 \sin^2\frac{1}{2}\theta}{Cm}$$