16.346 Astrodynamics
Fall 2008

Lecture 14 Hypergeometric Functions and Continued Fractions

John Wallis' Hypergeometric Series

$$a + a(a+b) + a(a+b)(a+2b) + \cdots + a(a+b)(a+2b)\ldots[a+(n-1)b] + \cdots$$

Hypergeometric Function

Named by Gauss' mentor Johann Pfaff 1765–1825

In the year 1812, Carl Friedrich Gauss published his book entitled:

GENERAL INVESTIGATIONS
CONCERNING THE INFINITE SERIES

$$1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}xx + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot 2\cdot 3\cdot\gamma(\gamma+1)(\gamma+2)}x^3 + \text{etc.}$$

We use the symbol $F(\alpha,\beta;\gamma;x)$ to represent this series.

Examples of Hypergeometric Functions

$$\log(1+x) = xF(1,1;2;-x)$$

$$\arctan x = xF\left(\frac{1}{2},1;\frac{3}{2};-x^2\right)$$

$$e^x = \lim_{\alpha\to\infty} F\left(\alpha,1;1;\frac{x}{\alpha}\right)$$

$$\sin x = \lim_{\substack{\alpha\to\infty\\ \beta\to\infty}} xF\left(\alpha,\beta;\frac{3}{2};-\frac{x^2}{4\alpha\beta}\right)$$

$$\cos x = \lim_{\substack{\alpha\to\infty\\ \beta\to\infty}} F\left(\alpha,\beta;\frac{1}{2};-\frac{x^2}{4\alpha\beta}\right)$$

Gauss' Differential Equation

$$x(1-x)\frac{d^2y}{dx^2} + [\gamma - (\alpha+\beta+1)x]\frac{dy}{dx} - \alpha\beta y = 0$$

has the general solution

$$y = c_1 F(\alpha,\beta;\gamma;x) + c_2 x^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;x)$$

Gauss' Continued Fraction Expansion

$$F_0 = F(\alpha,\beta;\gamma;x)$$

$$F_1 = F(\alpha,\beta+1;\gamma+1;x) \qquad F_1 - F_0 = \delta_1 x F_2 \qquad \delta_1 = \frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)}$$

$$F_2 = F(\alpha+1,\beta+1;\gamma+2;x) \qquad F_2 - F_1 = \delta_2 x F_3 \qquad \delta_2 = \frac{(\beta+1)(\gamma-\alpha+1)}{(\gamma+1)(\gamma+2)}$$

$$F_3 = F(\alpha+1,\beta+2;\gamma+3;x) \qquad F_3 - F_2 = \delta_3 x F_4 \qquad \delta_3 = \frac{(\alpha+1)(\gamma-\beta+1)}{(\gamma+2)(\gamma+3)}$$

$$F_4 = F(\alpha+2,\beta+2;\gamma+4;x) \qquad F_4 - F_3 = \delta_4 x F_5$$

$$G_0 = \frac{F_1}{F_0} \qquad G_0 - 1 = \delta_1 x G_1 G_0 \qquad G_0 = \frac{1}{1 - \delta_1 x G_1}$$

$$G_1 = \frac{F_2}{F_1} \qquad G_1 - 1 = \delta_2 x G_2 G_1 \qquad G_1 = \frac{1}{1 - \delta_2 x G_2}$$

$$G_2 = \frac{F_3}{F_2} \qquad G_2 - 1 = \delta_3 x G_3 G_2 \qquad G_2 = \frac{1}{1 - \delta_3 x G_3}$$

$$\frac{F(\alpha, \beta + 1; \gamma + 1; x)}{F(\alpha, \beta; \gamma; x)} = G_0 = \frac{1}{1 - \delta_1 x G_1} = \cfrac{1}{1 - \cfrac{\delta_1 x}{1 - \delta_2 x G_2}} = \cfrac{1}{1 - \cfrac{\delta_1 x}{1 - \cfrac{\delta_2 x}{1 - \delta_3 x G_3}}}$$

Since $F(\alpha, 0; \gamma; x) = 1$, we have developed a continued fraction expansion for

$$F(\alpha, 1; \gamma + 1; x)$$

Examples

$$\begin{aligned}
\log(1 + x) &= xF(1, 1; 2; -x) \\
\arctan x &= xF(\tfrac{1}{2}, 1; \tfrac{3}{2}; -x^2) \\
\arcsin x &= xF(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{3}{2}; x^2) = x\sqrt{1 - x^2}\, F(1, 1; \tfrac{3}{2}; x^2) \\
Q = \frac{2\psi - \sin 2\psi}{\sin^3 \psi} &= \frac{4}{3} F(3, 1; \tfrac{5}{2}; \sin^2 \tfrac{1}{2}\psi) \\
\operatorname{arctanh} x &= xF(\tfrac{1}{2}, 1; \tfrac{3}{2}; x^2)
\end{aligned}$$

Sufficient Conditions for Convergence of Continued Fractions

Class I

$$\cfrac{a_0}{b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}}}$$

Will either converge or oscillate between two different values.

$$\boxed{\lim_{n \to \infty} \frac{b_{n-1} b_n}{a_n} > 0}$$

Class II

$$\cfrac{a_0}{b_0 - \cfrac{a_1}{b_1 - \cfrac{a_2}{b_2 - \cfrac{a_3}{b_3 - \ddots}}}}$$

Will either converge or diverge to infinity.

$$\boxed{b_n \geq a_n + 1}$$

Note: All a_n and b_n are positive.

The Top-Down Method for Evaluating Continued Fractions — Pages 67–68

For a Class II continued fraction with $n = 1, 2, \ldots,$ we have

$$\delta_n = \cfrac{1}{1 - \dfrac{a_n}{b_{n-1}b_n}\delta_{n-1}} \qquad u_n = u_{n-1}(\delta_n - 1) \qquad \Sigma_n = \Sigma_{n-1} + u_n$$

where
$$\delta_0 = 1 \qquad u_0 = \Sigma_0 = \frac{a_0}{b_0}$$

Continued Fractions Versus Power Series

For the tangent function

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2,835}x^9 + \cdots = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{7 - \ddots}}}}$$

a. The series converges for $-\frac{1}{2}\pi \leq x \leq \frac{1}{2}\pi$.

b. The continued fraction converges for all x not equal to $\frac{1}{2}\pi \pm n\pi$.