16.323 Principles of Optimal Control
Spring 2008

April 19th, 2007

16.323 Midterm #2

This is a closed-book exam, but you are allowed 2 page of notes (both sides).

You have 1.5 hours.

There are three **3** questions with **values as given**.

Hint: To maximize your score, initially give a brief explanation of your approach before getting too bogged down in the equations.

1. (35pts) In the calculus of variations problem, where the goal is to minimize

$$J = \int_{t_0}^{t_f} g(x, \dot{x}, t)dt$$

we showed that the first order necessary condition is

$$\frac{\partial g}{\partial x} - \frac{d}{dt}\left\{\frac{\partial g}{\partial \dot{x}}\right\} = 0$$

subject to various (assumed to be well defined) initial and terminal conditions, depending on the problem statement. Now consider the following:

(a) If we can write $g \to g(\dot{x})$ (i.e. the function g is not an explicit function of x or time), show that there always exists a solution that is a linear function of time.

(b) If we can write $g \to g(x, t)$ (i.e. the function g is not an explicit function of $\dot{x}$), show that in general we would not expect a solution to exist.

(c) If we can write $g \to g(x, \dot{x})$ (i.e. the function g is not an explicit function of time), show that

$$g - \dot{x}\frac{\partial g}{\partial \dot{x}} = \text{constant}$$

2. (30pts) The dynamics of a reservoir system are given by the equations

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + u(t) & (1)\\ \dot{x}_2(t) &= x_1(t) & (2) \end{aligned}$$

where $x_1(t)$ and $x_2(t)$ correspond to the water height in each tank, and the inflow is constrained so that $0 \leq u(t) \leq 1$. Initially we have $x_1(0) = x_2(0) = 0$.

The objective is to maximize $x_2(1)$ subject to the constraint that $x_1(1) = 0.5$. Find the optimal input strategy $u(t)$ for the problem.

3. (35pts) Given the following plant dynamics,

$$\dot{x}(t) = 3x(t) + 2u(t) + 3w(t) \tag{3}$$
$$y(t) = 2x(t) + v(t) \tag{4}$$

where $w(t) \sim N(0,1)$ and $v(t) \sim N(0,4)$ are Gaussian, white noises.

(a) Find the steady-state gains and the closed-loop poles of the linear quadratic regulator that minimizes the following cost function,

$$J = \lim_{t_f \to \infty} \frac{1}{t_f} E\left\{ \frac{1}{2} \int_0^{t_f} (3x^2(t) + 4u^2(t))dt \right\}$$

(b) For this optimally-controlled system, what is the steady-state mean-squared value of $x(t)$?

(c) Given the plant dynamics above, find the steady-state gains and the closed-loop poles of the linear quadratic estimator for this system.

(d) The full stochastic linear optimal output feedback problem involves using $u(t) = -K\hat{x}(t)$. For this control policy, the compensator can be written,

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t) \tag{5}$$
$$u(t) = -C_c x_c(t) \tag{6}$$

In steady-state, what are A_c, B_c and C_c for this system?

(e) Write the closed-loop matrix A_{cl} for the combined plant and compensator dynamics $\begin{bmatrix} x \\ x_c \end{bmatrix}$. What are the closed loop eigenvalues of A_{cl} for this system?