

16.30/31 Homework Assignment #6

Goals: More on LQR; LQ servo; DOFB compensators; intro to LQ robustness

1. Consider the system

$$\begin{aligned} \dot{\mathbf{x}} &= (A + \Delta I_2)\mathbf{x} + Bu, \\ y &= C\mathbf{x}, \end{aligned} \quad A = \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [1 \ 0].$$

The ΔI_2 term corresponds to possible uncertainty in knowledge of the plant dynamics, where I_2 is the identity matrix. (You may use Matlab throughout this problem.)

In parts (a)-(c), we consider the *nominal* dynamics, i.e. $\Delta = 0$.

- Design an LQR controller using $Q = I_2$, $R = 0.01$, and implement the controller to perform reference tracking via $u = r - K\mathbf{x}$. Describe the tracking performance of this controller to a step input, and give the resulting steady-state error.
- Suppose we modify the control law to take the form $u = \bar{N}r - K\mathbf{x}$, where \bar{N} is a scalar. How should \bar{N} be chosen to minimize steady-state error? Verify that this results in good tracking performance for a step input.
- Implement an LQ servo controller for this system, using the weights

$$\tilde{Q} = \begin{bmatrix} Q & 0 \\ 0 & Q_I \end{bmatrix} \quad \text{and} \quad \tilde{R} = R$$

for the state and input, respectively, where Q and R are from part (a), and Q_I is chosen to achieve performance similar to part (b). Compare the step response performance for the controllers designed in parts (b) and (c).

- Suppose that in the actual system, $\Delta = -0.1$; compare and explain the step response performance of the controllers designed in parts (b) and (c).

2. Consider the system with transfer function

$$G(s) = \frac{s + 8}{(s + 1)(s + 10)}.$$

- Design a compensator using classical techniques that results in a dominant response with 10-90% rise time of about 0.8 seconds, and a peak overshoot between 0.5% and 1.5%.

- (b) Develop a state-space model of $G(s)$, and design a dynamic output feedback controller using the techniques discussed in class:
- (i) Use Ackermann's formula to place the regulator poles to meet the design requirements given above (should be near $-3 \pm 2j$). Show your calculations by hand and confirm in Matlab.
 - (ii) Use the estimator equivalent of Ackermann's formula to place the estimator poles such that the real part is twice that of the regulator poles. Show your calculations by hand and confirm in Matlab.
 - (iii) Combine the regulator and estimator to form the compensator $G_c(s)$. Does the closed-loop system meet the specs?
- (c) Compare your designs from part (a) and (b), using both the frequency and time responses.
3. One particularly useful modification of the standard LQR cost function is

$$J' = \int_0^{\infty} e^{2\alpha t} [\mathbf{x}^T \mathbf{x} + \rho \mathbf{u}^T \mathbf{u}] dt. \quad (\rho > 0, \alpha \geq 0)$$

The $e^{2\alpha t}$ term magnifies the weighting on the state and control deviations that occur later in the system response, compelling the system to respond more quickly. Interestingly enough, the only modification necessary to find the LQR controller which minimizes J' is to replace A with $A + \alpha I$ in the algebraic Riccati equation.

In this problem, we explore this LQR formulation for the first-order unstable system

$$\dot{x}(t) = ax(t) + u(t). \quad (a > 0)$$

- (a) Analytically design the controller $u = -Kx$ which minimizes J' , in terms of ρ , a , and α .
- (b) Identify the closed-loop pole(s) using this controller; prove that this controller guarantees exponential decay in the closed-loop system with the decay constant exceeding α .
- (c) What happens to the gain K and closed-loop pole(s) as $\rho \rightarrow 0$? As $\rho \rightarrow \infty$?

The remainder of this problem is **(16.31 required / 16.30 extra credit)**.

- (d) Draw a Nyquist plot of the loop transfer function $T(s) = K(sI - A)^{-1}B$ for some choice of a and α , and three very different values of ρ .
- (e) Use the Nyquist Stability Theorem to prove that the closed-loop system is stable *for any choice of* $a > 0$, $\rho > 0$, and $\alpha \geq 0$. Note that this includes the case $\alpha = 0$, which reduces to the conventional LQR problem.
Hints: You can prove that $T(j\omega)$ never penetrates the unit circle centered at -1 for any $\omega \in \mathbb{R}$. Consider $T(j0)$ and $T(\pm j\infty)$.

Later in the semester, we will revisit classical stability analysis to evaluate the robustness of LQR controllers and DOFB compensators, in general.

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