16.21 Techniques of Structural Analysis and Design
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Unit #5 - Constitutive Equations

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Constitutive Equations

For elastic materials:

$$\sigma_{ij} = \sigma_{ij}(\epsilon) = \rho \frac{\partial \widehat{U}_0}{\partial \epsilon_{ij}} \tag{1}$$

If the relation is linear:

$$\boxed{\sigma_{ij} = C_{ijkl}\epsilon_{kl}}, \text{ Generalized Hooke's Law} \tag{2}$$

In this expression: C_{ijkl} fourth-order tensor of material properties or *Elastic moduli* (How many material constants?). Making use of the symmetry of the stress tensor:

$$\sigma_{ij} = \sigma_{ji} \; \Rightarrow C_{jikl} = C_{ijkl} \tag{3}$$

Proof by (generalizable) example:

$$\begin{gathered}
\sigma_{21} = C_{21kl}\epsilon_{kl}, \; \sigma_{12} = C_{12kl}\epsilon_{kl} \\
\sigma_{21} = \sigma_{12} \; \Rightarrow \; C_{21kl}\epsilon_{kl} = C_{12kl}\epsilon_{kl} \\
\big(C_{21kl} - C_{12kl}\big)\epsilon_{kl} = 0 \; \Rightarrow \\
C_{21kl} = C_{12kl}
\end{gathered}$$

which generalizes to the statement. This reduces the number of material constants from 81 to 54. In a similar fashion we can make use of the symmetry of the strain tensor

$$\epsilon_{ij} = \epsilon_{ji} \ \Rightarrow C_{ijlk} = C_{ijkl} \tag{4}$$

This further reduces the number of material constants to 36. To further reduce the number of material constants consider the conclusion from the first law for elastic materials, equation (1):

$$\sigma_{ij} = \frac{\partial U_0}{\partial \epsilon_{ij}},\ U_0 : \text{strain energy density per unit volume} \tag{5}$$

$$C_{ijkl}\epsilon_{kl} = \frac{\partial U_0}{\partial \epsilon_{ij}} \tag{6}$$

$$\frac{\partial}{\partial \epsilon_{mn}}\left(C_{ijkl}\epsilon_{kl}\right) = \frac{\partial^2 U_0}{\partial \epsilon_{mn} \partial \epsilon_{ij}} \tag{7}$$

$$C_{ijkl}\delta_{km}\delta_{ln} = \frac{\partial^2 U_0}{\partial \epsilon_{mn} \partial \epsilon_{ij}} \tag{8}$$

$$C_{ijmn} = \frac{\partial^2 U_0}{\partial \epsilon_{mn} \partial \epsilon_{ij}} \tag{9}$$

Assuming equivalence of the mixed partials:

$$C_{ijkl} = \frac{\partial^2 U_0}{\partial \epsilon_{kl} \partial \epsilon_{ij}} = \frac{\partial^2 U_0}{\partial \epsilon_{ij} \partial \epsilon_{kl}} = C_{klij} \tag{10}$$

This further reduces the number of material constants to 21. The most general anisotropic linear elastic material therefore has 21 material constants. We are going to adopt *Voigt's notation*:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & symm & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix} \tag{11}$$

When the material has symmetries in its structure the number of material constants is reduced even further (see Unified treatment of this material). We are going to concentrate on the isotropic case:

Isotropic linear elastic materials

Most general isotropic 4th order isotropic tensor:

$$C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu\big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\big) \tag{12}$$

Replacing in:

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} \tag{13}$$

gives:

$$\sigma_{ij} = \lambda \delta_{ij}\epsilon_{kk} + \mu\big(\epsilon_{ij} + \epsilon_{ji}\big) \tag{14}$$

$$\boxed{\sigma_{ij} = \lambda \delta_{ij}\epsilon_{kk} + \mu\big(\epsilon_{ij} + \epsilon_{ji}\big)} \tag{15}$$

Examples

$$\sigma_{11} = \lambda \delta_{11}\big(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}\big) + \mu\big(\epsilon_{11} + \epsilon_{11}\big) = \big(\lambda + 2\mu\big)\epsilon_{11} + \lambda\epsilon_{22} + \lambda\epsilon_{33} \tag{16}$$

$$\sigma_{12} = 2\mu\epsilon_{12} \tag{17}$$

Practice problem: Write the matrix of coefficients C (elastic moduli) for an isotropic material (Voigt form) in Mathematica.

Compliance matrix for an isotropic elastic material

From experiments one finds:

$$\epsilon_{11} = \frac{1}{E}\Big[\sigma_{11} - \nu\big(\sigma_{22} + \sigma_{33}\big)\Big] \tag{18}$$

$$\epsilon_{22} = \frac{1}{E}\Big[\sigma_{22} - \nu\big(\sigma_{11} + \sigma_{33}\big)\Big] \tag{19}$$

$$\epsilon_{33} = \frac{1}{E}\Big[\sigma_{33} - \nu\big(\sigma_{11} + \sigma_{22}\big)\Big] \tag{20}$$

$$2\epsilon_{23} = \frac{\sigma_{23}}{G}, \quad 2\epsilon_{13} = \frac{\sigma_{13}}{G}, \quad 2\epsilon_{12} = \frac{\sigma_{12}}{G} \tag{21}$$

In these expressions, E is the Young's Modulus, ν the Poisson's ratio and G the shear modulus. They are referred to as the *engineering constants*, since they are obtained from experiments. In Unified we demonstrated that $G = \frac{E}{2(1+\nu)}$. This expressions can be written in the following matrix form:

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ & & \frac{1}{E} & 0 & 0 & 0 \\ & & & \frac{1}{G} & 0 & 0 \\ & symm & & & \frac{1}{G} & 0 \\ & & & & & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} \tag{22}$$

Invert and compare with:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda + 2\mu & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & symm & & & \mu & \\ & & & & & \mu \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix} \tag{23}$$

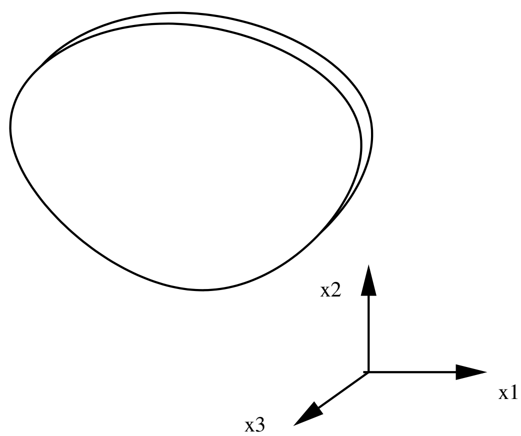
and conclude that:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = G \tag{24}$$

Plane stress

Consider situations in which:

$$\sigma_{i3} = 0 \tag{25}$$



Then:

$$\epsilon_{11} = \frac{1}{E}\big(\sigma_{11} - \nu\sigma_{22}\big) \tag{26}$$

$$\epsilon_{22} = \frac{1}{E}\big(\sigma_{22} - \nu\sigma_{11}\big) \tag{27}$$

$$\epsilon_{33} = \frac{-\nu}{E}\big(\sigma_{11} + \sigma_{22}\big) \neq 0\,!!! \tag{28}$$

$$\epsilon_{23} = \epsilon_{13} = 0 \tag{29}$$

$$\epsilon_{12} = \frac{\sigma_{12}}{2G} = \frac{(1+\nu)\sigma_{12}}{E} \tag{30}$$

In matrix form:

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} \tag{31}$$

Inverting gives the relations among stresses and strains for *plane stress*:

$$\boxed{\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{(1-\nu)}{2}\end{bmatrix}\begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ 2\epsilon_{12}\end{bmatrix}} \tag{32}$$

Plane strain

In this case we consider situations in which:

$$\epsilon_{i3} = 0 \tag{33}$$

Then:

$$\epsilon_{33} = 0 = \frac{1}{E}\Big[\sigma_{33} - \nu\big(\sigma_{11} + \sigma_{22}\big)\Big], \text{ or:} \tag{34}$$

$$\sigma_{33} = \nu\big(\sigma_{11} + \sigma_{22}\big) \tag{35}$$

$$\begin{aligned}\epsilon_{11} &= \frac{1}{E}\Big\{\sigma_{11} - \nu\big[\sigma_{22} + \nu\big(\sigma_{11} + \sigma_{22}\big)\big]\Big\}\\ &= \frac{1}{E}\Big[\big(1-\nu^2\big)\sigma_{11} - \nu\big(1+\nu\big)\sigma_{22}\Big]\end{aligned} \tag{36}$$

$$\epsilon_{22} = \frac{1}{E}\Big[\big(1-\nu^2\big)\sigma_{22} - \nu\big(1+\nu\big)\sigma_{11}\Big] \tag{37}$$

In matrix form:

$$\begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ 2\epsilon_{12}\end{bmatrix} = \frac{1}{E}\begin{bmatrix}1-\nu^2 & -\nu(1+\nu) & 0\\ -\nu(1+\nu) & 1-\nu^2 & 0\\ 0 & 0 & 2(1+\nu)\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} \tag{38}$$

Inverting gives the relations among stresses and strains for *plane strain*:

$$\boxed{\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{(1-2\nu)}{2}\end{bmatrix}\begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ 2\epsilon_{12}\end{bmatrix}} \tag{39}$$

Practice problem: Verify equations (32) and (39) using Mathematica.

0.0.1 Thermal strains

We are going to consider the strains produced by changes of temperature (ϵ^{θ}). These strains have inherently a dilatational nature (thermal expansion or contraction) and do not cause any shear. Thermal strains are proportional to temperature changes. For isotropic materials:

$$\epsilon_{ij}^{\theta} = \alpha \Delta\theta \delta_{ij} \tag{40}$$

The total strains (ϵ_{ij}) are then due to the (additive) contribution of the *mechanical strains* (ϵ_{ij}^{M}), i.e., those produced by the stresses and the thermal strains:

$$\epsilon_{ij} = \epsilon_{ij}^{M} + \epsilon_{ij}^{\theta} \tag{41}$$

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}^{M} = C_{ijkl}(\epsilon_{kl} - \epsilon_{kl}^{\theta}), \quad \text{or} \tag{42}$$

$$\boxed{\sigma_{ij} = C_{ijkl}(\epsilon_{kl} - \alpha\Delta\theta\delta_{kl})} \tag{43}$$

Practice problem: Write the relationship between stresses and strains for an isotropic elastic material whose Lamé constants are λ and μ and whose coefficient of thermal expansion is α. constants and

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