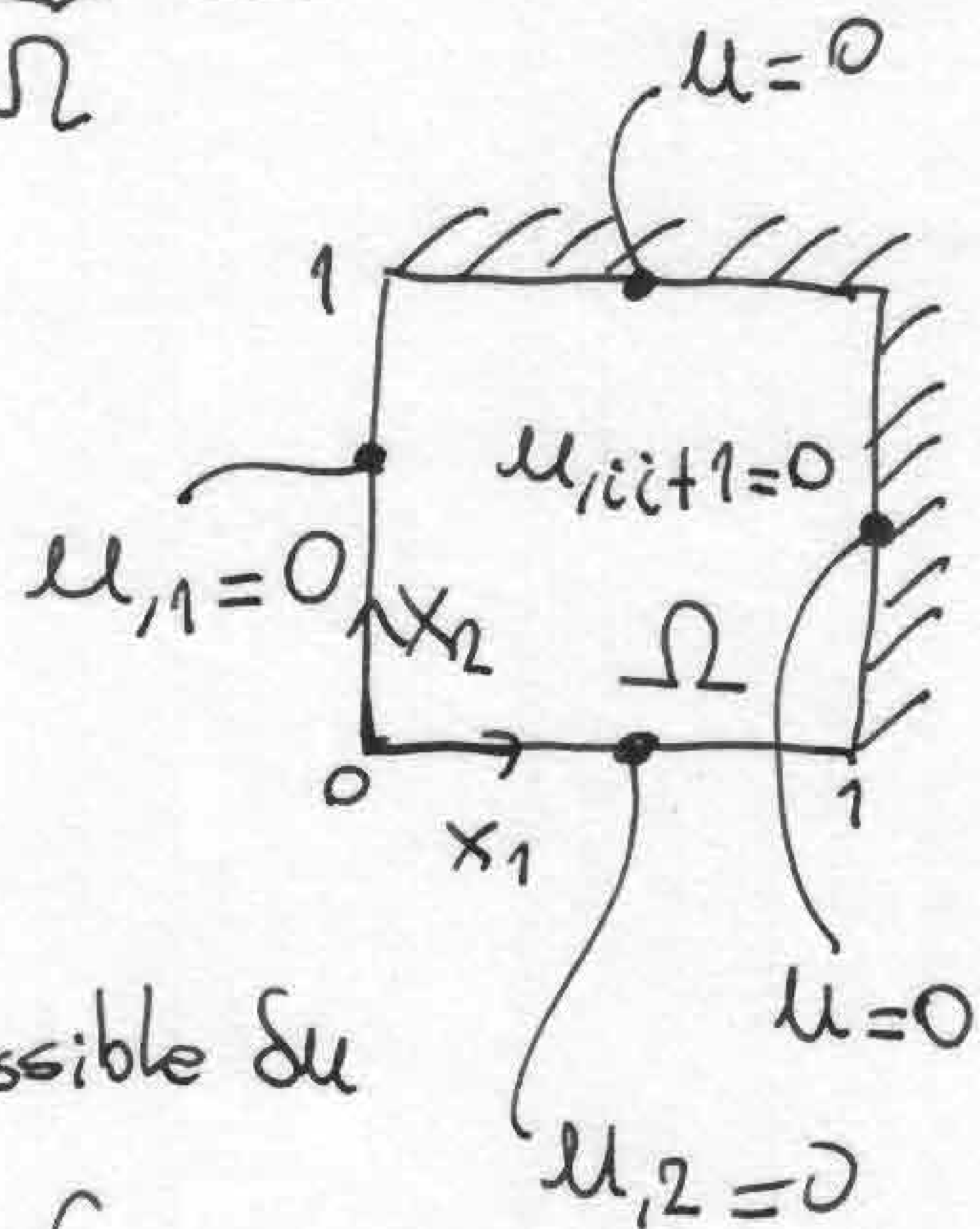
(7.47)

$$u_{,ii} + 1 = 0 \quad \text{in} \quad \underbrace{[0,1]\times[0,1]}_{\Omega}$$

$$u(1, x_2) = u(x_1, 1) = 0$$

$$u_{,1}(0, x_2) = u_{,2}(x_1, 0) = 0$$



<u>Weak form:</u>

$$\int_\Omega (u_{,ii} + 1)\,\delta u \, d\Omega = 0$$

$\forall$ admissible δu

$$-\int_\Omega u_{,i}\,\delta u_{,i}\, d\Omega + \underbrace{\int_\Omega (u_{,i}\,\delta u)_{,i}\, d\Omega}_{\displaystyle\int_{\partial\Omega} u_{,i}\,\delta u\, n_i\, d\partial\Omega} + \int_\Omega 1\,\delta u\, d\Omega = 0$$

$\hookrightarrow$ boundary of Ω

Let's analyze the boundary integral:

$$\int_{\partial\Omega} u_{,i}\,\delta u\, n_i\, d\partial\Omega = \int_0^1 (u_{,1}\,\delta u\, n_1 + u_{,2}\,\delta u\, n_2)\Big|_{(x_1,0)} dx_1 +$$

$$\int_0^1 (\qquad\qquad)\Big|_{(1,x_2)} dx_2 +$$

$$\int_1^0 (\qquad\qquad)\Big|_{(x_1,1)} (-dx_1) +$$

$$\int_1^0 (\qquad\qquad)\Big|_{(0,x_2)} (-dx_2)$$

It can be seen that in the essential part of the boundary the integrand vanishes due to the admissibility requirement on δu, i.e., δu must satisfy the homogeneous essential boundary condition:

$$\delta u = 0 \quad \text{where "} u \text{" is specified}$$

$$\delta u(1, x_2) = \delta u(x_1, 1) = 0 \quad \text{(boundary)}$$

Therefore, the second and third integrals vanish.
In the first integral, $n_1 = 0$, so the first term $= 0$,
$n_2 = -1$ but $u_{,2}(x_1, 0) = 0$, so the integral vanishes.
A similar argument shows that the fourth integral $= 0$
Therefore,
$$\int_{\partial\Omega} u_{,i}\, \delta u\, n_i \, d\partial\Omega = 0$$

The weak form is:

$$\int_{\Omega} u_{,i}\, \delta u_{,i}\, d\Omega = \int_{\Omega} 1\, \delta u\, d\Omega \qquad \forall \text{ admissible } \delta u$$

This is taken as a basis for approximation:

$$\underset{(x_1,x_2)}{u} \sim \underset{(x_1,x_2)}{u_h} = \underset{(x_1,x_2)}{\phi_a} U_a \quad , \quad \delta u(x_1,x_2) \sim \delta u_h(x_1,x_2) = \phi_a(x_1,x_2)\, \delta U_a$$

Inserting in weak form:

$$\delta U_a \underbrace{\int_\Omega \phi_{b,i}\, \phi_{a,i}\, d\Omega}_{K_{ab}} \underbrace{U_b}_{U_b} = \delta U_a \underbrace{\int_\Omega 1\, \phi_a\, d\Omega}_{R_a} \quad \forall \text{ admissible } \delta U_a$$

Choose appropriate shape functions and solve system:

$$\phi_1(x_1, x_2) = \cos\left(\frac{\pi}{2}\frac{x_1}{1}\right) \cos\left(\frac{\pi}{2}\frac{x_2}{1}\right)$$

which leads to admissible δu_h, $\forall \delta U_a$, i.e.

$$\phi_1(1, x_2) = \overbrace{\cos\frac{\pi}{2}}^{0} \cos\left(\frac{\pi}{2}\frac{x_2}{1}\right) = 0 \Rightarrow$$

$$\delta u_h(1, x_2) = \phi_1\, \delta U_1 = 0$$

$$\phi_{1,1} = -\frac{\pi}{2} \sin\left(\frac{\pi}{2} x_1\right) \cos\left(\frac{\pi}{2}\frac{x_2}{1}\right)$$

$$\phi_{1,2} = -\frac{\pi}{2} \cos\left(\frac{\pi}{2} x_1\right) \sin\left(\frac{\pi}{2} x_2\right)$$

$$K_{11} = \int_{-1}^{1}\int_{-1}^{1} \left(\phi_{1,1}^2 + \phi_{1,2}^2\right) dx_1\, dx_2$$

solution in Mathematica

$$R_1 = \int_{-1}^{1}\int_{-1}^{1} 1\ \phi_1\ dx_1\, dx_2$$