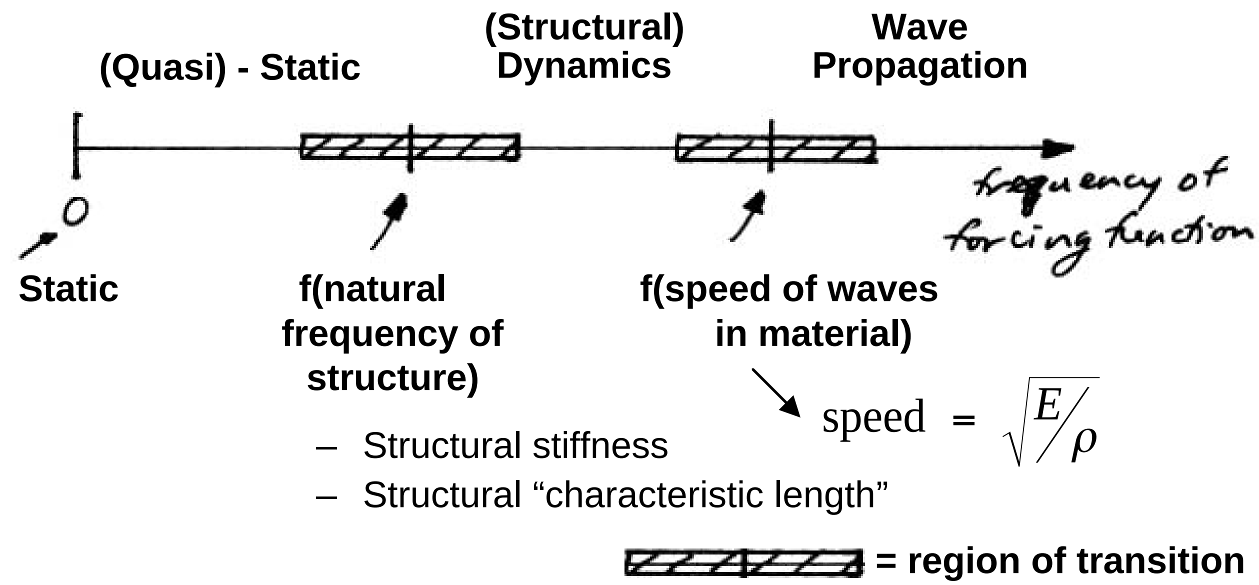
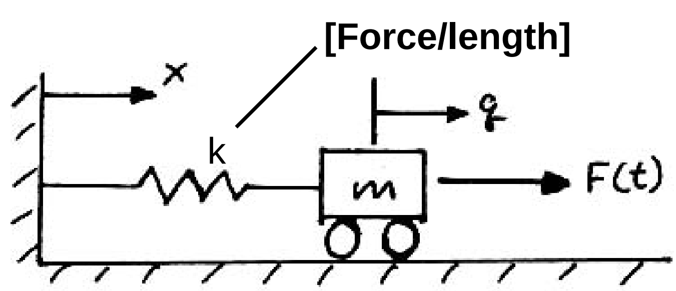
16.20 HANDOUT #6
Fall, 2002
(Introduction To) Structural Dynamics

REGIMES



SPRING-MASS SYSTEMS



Static equation: F = kq $\qquad \dot{(\ )} = \frac{d}{dt}$

Dynamic: $m\ddot{q} + kq = F(t)$ (No damping)

$m\ddot{q} + c\dot{q} + kq = F(t)$ (With damping)

Inertial Load = – (mass) x (acceleration)

General form:

$\underset{\sim}{m}\underset{\sim}{\ddot{q}} + \underset{\sim}{k}\underset{\sim}{q} = \underset{\sim}{F}$

$\underset{\sim}{m}$ = mass matrix
$\underset{\sim}{k}$ = stiffness matrix

or

$$m_{ij}\,\ddot{q}_j + k_{ij}\,q_j = F_i$$

$\underset{\sim}{F}$ = force vector
$\underset{\sim}{q}$ = d. o. f. vector
d. o. f. = degree of freedom

i, j = 1, 2…n

n = number of degrees of freedom of system

FREE VIBRATION

$$q(t) = C_1' \sin \omega t + C_2' \cos \omega t$$

general solution for single spring-mass system

natural frequency: $\omega = \sqrt{\dfrac{k}{m}}$

FORCED VIBRATION

dirac delta function: $\delta(t - \tau) = 0 \quad @ \quad t \neq \tau$

$$\delta(t - \tau) \to \infty \quad @ \quad t = \tau$$

$$\int_{-\infty}^{\infty} \delta(t - \tau)\, dt = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} g(t)\, \delta(t - \tau)\, dt = g(\tau)$$

unit impulse response:

$$q(t) = \begin{cases} \dfrac{1}{m\,\omega} \sin \omega\,(t - \tau) & \text{for } t \geq \tau \\ 0 & \text{for } t \leq \tau \end{cases}$$

Duhamel's (convolution) integral:

$$q(t) = \int_0^t F(\tau)\, h(t - \tau)\, d\tau$$

Response to arbitrary f(t):

$$q(t) = \frac{1}{m\,\omega}\int_0^t F(\tau)\sin\omega\,(t-\tau)\;d\tau + \frac{\dot{q}(0)}{\omega}\sin\omega t + q(0)\cos\omega t$$

Sinusoidal force:

$$F(t) = F_o \sin\Omega t$$

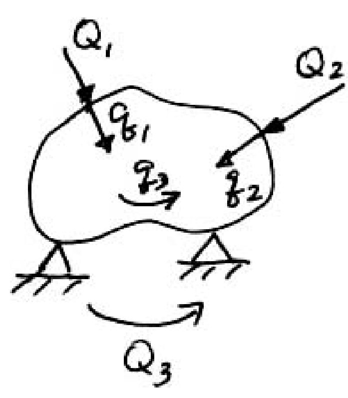
$$q(t) = \underbrace{C_1 \sin\omega t + C_2 \cos\omega t}_{\text{Starting transient}} + \underbrace{\frac{F_o}{k\left(1 - \frac{\Omega^2}{\omega^2}\right)}\sin\Omega t}_{\text{Steady state response}}$$

Dynamic Magnification Factor = $\dfrac{1}{\left(1 - \frac{\Omega^2}{\omega^2}\right)}$

•Note: resonance as $\Omega \rightarrow \omega$, (DMF $\rightarrow \infty$)

INFLUENCE COEFFICIENTS

q_i = generalized displacement
Q_i = generalized force



$$q_i = C_{ij}\,Q_j$$

C_{ij} = Flexibility Influence Coefficient

Maxwell's Theorem of Reciprocal Deflection: $C_{ij} = C_{ji}$

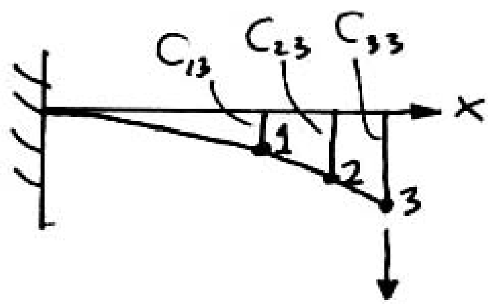
$$Q_i = \left[k_{ij}\right]\{q_j\}$$ k_{ij} = Stiffness Influence Coefficient

where: $k_{ij} = k_{ji}$

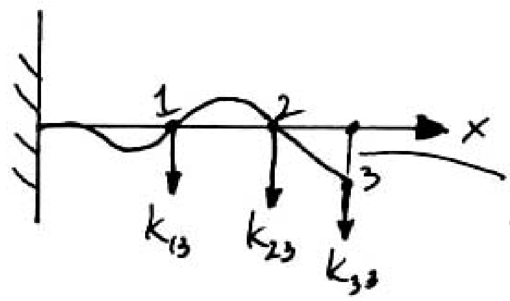
$$\left[k_{ij}\right] = \left[C_{ij}\right]^{-1}$$

Physical Interpretations

C_{ij} = displacement at i due to unit load at j



k_{ij} = force at i due to unit displacement at j and with other displacements equal to 0



MULTI DEGREE-OF-FREEDOM SYSTEMS

$$\underset{\sim}{m}\ddot{\underset{\sim}{q}} + \underset{\sim}{k}\underset{\sim}{q} = \underset{\sim}{F}$$

$$\left|\underset{\sim}{k} - \omega^2\underset{\sim}{m}\right| = 0$$

ω - eigenvalues = natural frequency

$\underset{\sim}{\phi}^{(r)}$ - eigenvector = mode

\# of eigenvalues = # of degrees of freedom

$$\underset{\sim}{q}_{i\hom} = \underset{\sim}{\phi}_i^{(r)} e^{i\omega_r t} = C_1\underset{\sim}{\phi}_i^{(r)} \sin\omega_r t + C_2\underset{\sim}{\phi}_i^{(r)} \cos\omega_r t$$

--> orthogonality of modes

$$\underset{\sim}{\phi}^{(r)T} \underset{\sim}{m}\, \underset{\sim}{\phi}^{(s)} = \delta_{rs} M_r \begin{cases} \underset{\sim}{\phi}^{(r)T} \underset{\sim}{m}\, \underset{\sim}{\phi}^{(s)} = 0 & \text{for r} \neq \text{s} \\ \underset{\sim}{\phi}^{(r)T} \underset{\sim}{m}\, \underset{\sim}{\phi}^{(r)} = M_r & \text{for r = s} \end{cases}$$

Kronecker delta:

$\delta_{rs} = 0$ for r ≠ s
$\delta_{rs} = 1$ for r = s

<u>NORMAL EQUATIONS OF MOTION</u> (discrete system)

$$\underset{\sim}{q} = \underset{\sim}{\phi}\, \underset{\sim}{\xi}$$

$$\Rightarrow M_r \ddot{\xi}_r + M_r \omega_r^2 \xi_r = \Xi_r$$

Generalized mass of r[th] mode:

$$M_r = \underset{\sim}{\phi}^{(r)^T} \underset{\sim}{m}\, \underset{\sim}{\phi}^{(r)} = \left[\phi_1^{(r)} \quad \phi_2^{(r)} \quad \cdots \right] m \begin{Bmatrix} \phi_1^{(r)} \\ \phi_2^{(r)} \\ \vdots \end{Bmatrix}$$

Generalized force of the r[th] mode:

$$\Xi_r = \underset{\sim}{\phi}^{(r)^T} \underset{\sim}{F} = \left[\phi_1^{(r)} \quad \phi_2^{(r)} \quad \cdots \right] \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \end{Bmatrix}$$

$\underset{\sim}{\xi}$ = normal coordinates

Homogeneous Solution:

$$\xi_r = a_r \sin\omega_r t + b_r \cos\omega_r t$$

$$\xi_r(0) = \frac{1}{M_r} \underset{\sim}{\phi}^{(r)T} \underset{\sim}{m}\, q_i(0) = b_r$$

$$\dot{\xi}_r(0) = \frac{1}{M_r} \underset{\sim}{\phi}^{(r)T} \underset{\sim}{m}\, \dot{q}_i(0) = a_r \omega_r$$

Particular Solution:

$$\xi_r(t) = \frac{1}{M_r \omega_r} \int_0^t \Xi_r(\tau) \sin \omega_r (t - \tau)\, d\tau$$

CONTINUOUS SYSTEMS

$$w(x,t) = \overline{w}(x)\, e^{i\omega t}$$

$$\overline{w}(x) = e^{px}$$ (homogeneous solution)

Beam-column

- axial force = 0
- cross-section constant

$$EI \frac{d^4 w}{dx^4} + m\ddot{w} = 0$$

$$\overline{w}(x) = C_1 \sinh \lambda x + C_2 \cosh \lambda x + C_3 \sin \lambda x + C_4 \cos \lambda x$$

with: $$\lambda = \left(\frac{m\omega^2}{EI}\right)^{1/4}$$

--> orthogonality of modes

$$\int_0^l m(x)\, \phi_r(x)\, \phi_s(x)\, dx = M_r\, \delta_{rs} \begin{cases} = 0 \quad \text{for } r \neq s \\ M_r = \int_0^l m(x)\, \phi_r^{\,2}(x)\, dx \end{cases}$$

NORMAL EQUATIONS OF MOTION (continuous systems)

$$w(x,t) = \sum_{r=1}^{\infty} \phi_r(x)\, \xi_r(t)$$

$$\Rightarrow M_r \ddot{\xi}_r + M_r \omega_r^{\,2} \xi_r = \Xi_r$$

Generalized mass of rth mode: $M_r = \int_0^l m\, \phi_r^{\,2}\, dx$

Generalized force of rth mode: $\Xi_r = \int_0^l \phi_r\, p_z(x,t)\, dx$

$\xi_r^{(t)}$ = normal coordinates

Forced vibration (Particular Solution for r^{th} mode):

$$\xi_r(t) = \frac{1}{M_r \omega_r} \int_0^t \Xi_r(\tau)\, \sin \omega_r (t - \tau)\, d\tau$$