Unit 8
Solution Procedures

Readings:

R	Ch. 4
T & G	17, Ch. 3 (18-26)
	Ch. 4 (27-46)
	Ch. 6 (54-73)

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Summarizing what we've looked at in elasticity, we have:

15 equations	<u>in</u>	15 unknowns
- 3 equilibrium		- 6 strains
- 6 strain-displacement		- 3 displacements
- 6 stress-strain		- 6 stresses

These must be solved for a generic body under some generic loading *subject to* the prescribed **boundary conditions** (B.C.'s)

There are two types of boundary conditions:

1. Normal (stress prescribed)
2. Geometric (displacement prescribed)

--> *you must have one or the other*

To solve this system of equations subject to such constraints over the continuum of a generic body is, in general, quite a challenge. There are basically *two* solution procedures:

1. **Exact** -- satisfy all the equations and the B.C.'s
2. **Numerical** -- come as "close as possible" (energy methods, etc.)

Let's consider "exact" techniques. A common, and classic, one is:

<u>Stress Functions</u>

- Relate six stresses to (fewer) functions defined in such a manner that they identically satisfy the equilibrium conditon
- Can be done for 3-D case
- Can be done for anisotropic (most often orthotropic) case

--> See: Lekhnitskii, <u>Anisotropic Plates</u>, Gordan & Breach, 1968.

--> Let's consider

- plane stress
- (eventually) isotropic

8 equations	<u>in</u>	8 unknowns
- 2 equilibrium		- 3 strains
- 3 strain-displacement		- 2 displacements
- 3 stress-strain		- 3 stresses

Define the "Airy" Stress Function = $\phi(x, y)$

English mathematician *a scalar*

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} + V \qquad (8-1)$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} + V \qquad (8-2)$$

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \qquad (8-3)$$

where: V = potential function for body forces f_x and f_y

$$f_x = -\frac{\partial V}{\partial x} \qquad f_y = -\frac{\partial V}{\partial y}$$

V exists if

$$\nabla \times f = 0 \qquad \text{(curl)}$$

that is: $$\frac{\partial f_x}{\partial y} = \frac{\partial f_y}{\partial x}$$

Recall that curl f = 0 $\Rightarrow$ "conservative" field
- gravity forces
- spring forces
- etc.

What does that compare to in fluids?
Irrotational flow

Look at how ϕ has been defined and what happens if we place these equations (8-1 - 8-3) into the plane stress equilibrium equations:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + f_x = 0 \qquad \text{(E1)}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + f_y = 0 \qquad \text{(E2)}$$

we then get:

$$(E1): \quad \frac{\partial}{\partial x}\left(\frac{\partial^2 \phi}{\partial y^2} + V\right) + \frac{\partial}{\partial y}\left(-\frac{\partial^2 \phi}{\partial x \partial y}\right) - \frac{\partial V}{\partial x} = 0 \quad \textbf{?}$$

$$\Rightarrow \frac{\partial^3 \phi}{\partial x \partial y^2} + \frac{\partial V}{\partial x} - \frac{\partial^3 \phi}{\partial x \partial y^2} - \frac{\partial V}{\partial x} = 0 \qquad (\underline{\text{yes}})$$

$$(E2): \quad \frac{\partial}{\partial x}\left(-\frac{\partial^2 \phi}{\partial x \partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial^2 \phi}{\partial x^2} + V\right) - \frac{\partial V}{\partial y} = 0 \quad \textbf{?}$$

$$\Rightarrow \frac{\partial^3 \phi}{\partial x^2 \partial y} + \frac{\partial^3 \phi}{\partial x^2 \partial y} + \frac{\partial V}{\partial y} - \frac{\partial V}{\partial y} = 0 \qquad (\underline{\text{yes}})$$

$\Rightarrow$ **<u>Equilibrium</u> <u>automatically</u> <u>satisfied</u> using Airy stress function!**

Does that mean that any function we pick for ϕ (x, y) will be valid?

<u>No</u>, it will satisfy equilibrium, but we still have the strain-displacement and stress-strain equations. If we use these, we can get to the governing equation:

<u>Step 1</u>: Introduce ϕ into the stress-strain equations (compliance form):

$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy}\right) \qquad \text{(E3)}$$

$$\varepsilon_{yy} = \frac{1}{E}\left(-\nu\sigma_{xx} + \sigma_{yy}\right) \qquad \text{(E4)}$$

$$\varepsilon_{xy} = \frac{2\left(1 + \nu\right)}{E}\sigma_{xy} \qquad \text{(E5)}$$

So:

$$\varepsilon_{xx} = \frac{1}{E}\left(\frac{\partial^2\phi}{\partial y^2} - \nu\frac{\partial^2\phi}{\partial x^2}\right) + \frac{\left(1 - \nu\right)}{E}V \qquad \text{(E3}'\text{)}$$

$$\varepsilon_{yy} = \frac{1}{E}\left(\frac{\partial^2\phi}{\partial x^2} - \nu\frac{\partial^2\phi}{\partial y^2}\right) + \frac{\left(1 - \nu\right)}{E}V \qquad \text{(E4}'\text{)}$$

$$\varepsilon_{xy} = -\frac{2\left(1 + \nu\right)}{E}\frac{\partial^2\phi}{\partial x\partial y} \qquad \text{(E5}'\text{)}$$

<u>Step 2</u>: Use these in the plane stress compatibility equation:

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \qquad \text{(E6)}$$

$\Rightarrow$ we get quite a mess! After some rearranging and manipulation, this results in:

$$\boxed{\begin{aligned} &\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^2 \phi}{\partial y^4} = \\ &\underbrace{-E\alpha\left\{\frac{\partial^2 (\Delta T)}{\partial x^2} + \frac{\partial^2 (\Delta T)}{\partial y^2}\right\}}_{\text{temperature term we haven't yet considered}} - (1 - \nu)\left\{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right\} \end{aligned}} \qquad (*)$$

α = coefficient of thermal expansion
ΔT = temperature differential

This is the basic equation for isotropic plane stress in Stress Function form

<u>Recall</u>: ϕ is a scalar

If we recall a little mathematics, the <u>Laplace</u> <u>Operator</u> in 2-D is:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\Rightarrow \nabla^2\nabla^2 = \nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

This is the <u>biharmonic</u> <u>operator</u> (also used in fluids)

So the (*) equation can be written:

$$\boxed{\nabla^4\phi = -E\alpha\nabla^2(\Delta T) - (1 - \nu)\nabla^2 V} \quad (*)$$

Finally, in the absence of temperature effects and body forces this becomes:

$$\nabla^4\phi = 0$$ ***<u>homogeneous</u> <u>form</u>***

What happened to E, ν??

$\Rightarrow$ this function, and accompanying governing equation, could be defined in any curvilinear system (we'll see one such example later) and in plane strain as well.

But…what's this all useful for???

This may all seem like "magic". Why were the σ's assumed as they were?

This is not a direct solution to a posed problem, *per se*, but is known as…

The Inverse Method

In general, for cases of plane stress without body force or temp ($\nabla^4 \phi = 0$):

1. A stress function ϕ (x, y) is assumed that satisfies the biharmonic equation
2. The stresses are determined from the stress function as defined in equations (8-1) - (8-3)
3. Satisfy the boundary conditions (of applied tractions)
4. ★★ Find the (*structural*) problem that this satisfies

- Mathematicians actually did this and created many solutions. So there are many stress functions that have been found to solve specific structural problems

  (see, for example, Rivello, pp, 72-73 also T & G)

- These are linear solutions and thus the “Principle of Superposition” applies such that these can be combined to solve any particular problem
- The inverse method yields an <u>exact</u> <u>solution</u>. In real life, an <u>exact</u> solution generally cannot be obtained. We often “notch it down one” and resort to the…

<u>The Semi-Inverse Method</u>

This is basically the same as the Inverse Method except that the solution is not exact in that we…

- Make simplifying assumptions to get solvable equations. These can be with regard to:
  - stress components
  - displacement components

- Assumptions are based on physical intuition, experimental evidence, prior experience (sheer need?)
- Assumptions may be due to…
  - – boundary conditions not properly met
  - – etc.

(Note: plane stress is generally an assumption)

⇒ There is an important concept that allows us to make such assumptions:

St. Venant's Principle

"If the loading distribution on the small section of the surface of an elastic body is replaced by another loading which has the same resultant force and moment as the original loading, then no appreciable changes will occur in the stresses in the body except in the region near the surface where the loading is altered".

What does this really say?

⇒"Far" from the specifics of load introduction / boundary conditions, the specifics of such are unimportant

- This is a ramification of the issue of "scale" as discussed earlier in this course and in Unified.

- ("Rule of Thumb") Generally, the area where "specifics" are important extends into the body for a distance equal to about the greatest linear dimension of the portion of the surface on which the loading / B.C. occurs.
- This allows us to get solutions for most parts of a structure via such a method.

But failure often originates/occurs in a region of load introduction/boundary condition

(example: where do nailed/screwed boards break?)

Examples (for stress functions)

Example 1 (assume $\Delta T = 0$, $V = 0$ [no body forces])

Pick $\phi = C_1 y^2$

this satisfies $\nabla^4 \phi = 0$

C_1 is a constant…will be determined by satisfying the B.C.'s

Using the definitions:

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = 2C_1$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = 0$$

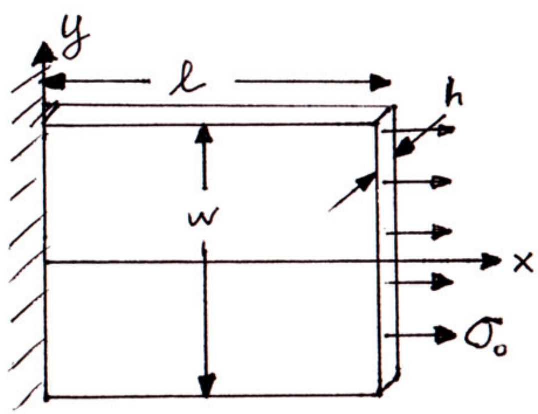
$$\sigma_{xy} = 0$$

gives the state of stress

What problem does this solve???

<u>Uniaxial</u> <u>loading</u>

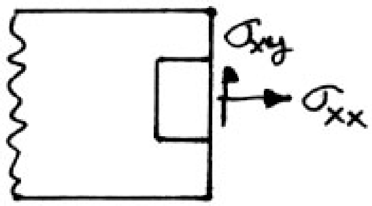
Figure 8.1 **Representation of Uniaxial Plane Stress Loading**



applied stress (uniform stress on end)

Check the B.C.'s:

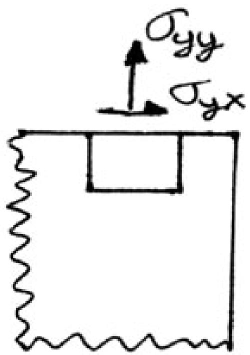
@ $x = \ell, 0$



$$\sigma_{xx} = \sigma_o = 2C_1 \Rightarrow C_1 = \frac{\sigma_o}{2}$$

$$\sigma_{xy} = 0$$

@ $y = \pm \frac{w}{2}$



$$\sigma_{yy} = 0$$

$$\sigma_{xy} = 0$$

Thus: $\phi = \frac{\sigma_o}{2} y^2$

- The strains can then be evaluated using the stress-strain equations (compliance form).
- The displacements can then be determined by using the strain-displacement relations and integrating and applying the displacement B.C.'s.

In this case, you get:

$$u = \frac{\sigma_o}{E} x$$

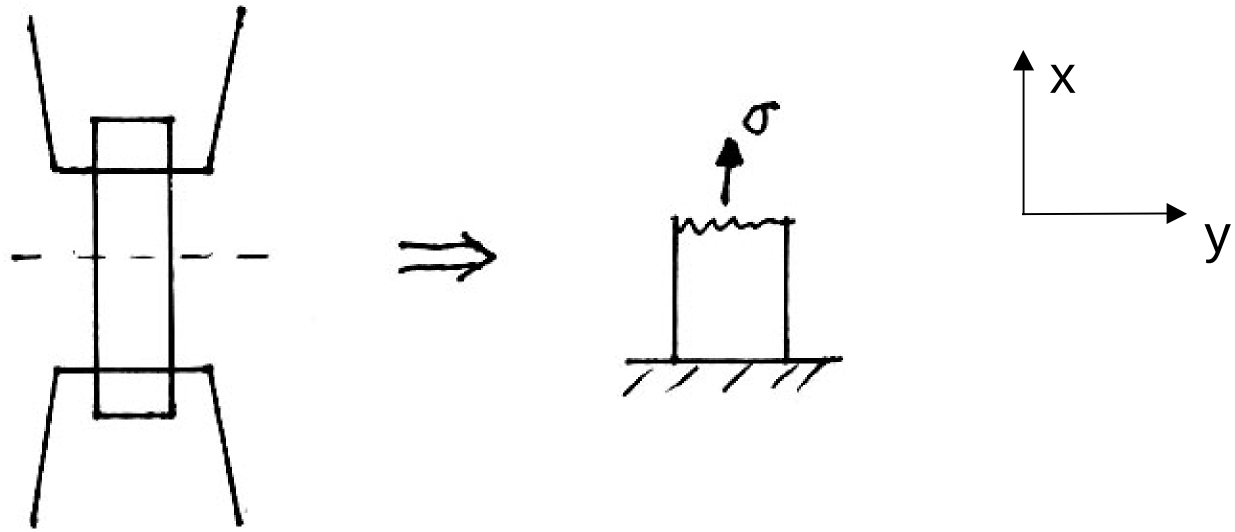
$$v = -\frac{\nu}{E} \sigma_o y$$

(**<u>Note</u>:** elastic constants now come in)

--> Let's look at the "real" case and see where/why we have to apply St. Venant's Principle and the "Semi - Inverse Method".

Consider a test coupon in mechanical grips (end of specimen is fixed):

Figure 8.2 **Representation of uniaxial test specimen and resultant stress state**



These grips allow <u>no</u> displacement in the y-direction

--> v = 0 both @ x = 0, ℓ

The solution gives:

$$v = \frac{-\nu\sigma}{E} y \neq 0 \text{ in general @ } x = 0, \ell$$

<u>But</u>, "far" from the effects of load introduction, the solution ϕ holds.

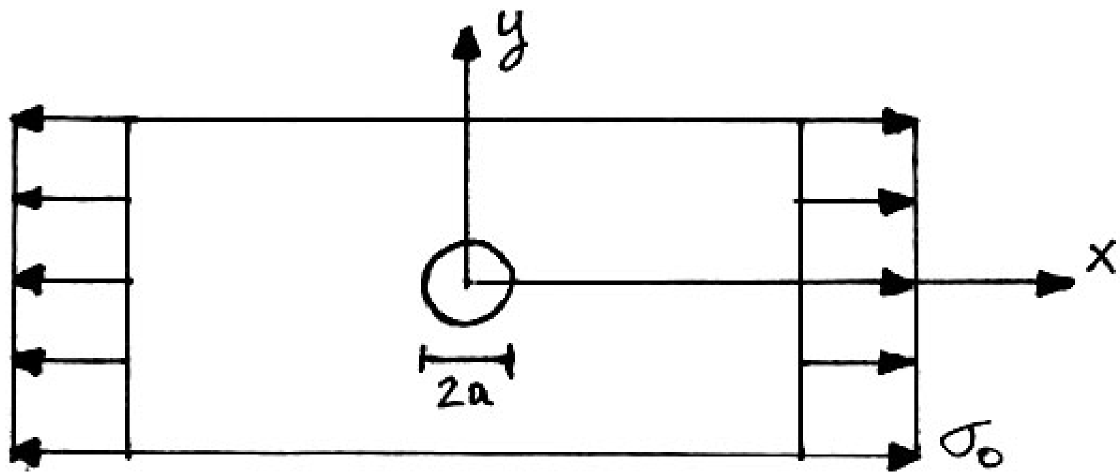
Near the grips, biaxial stresses arise. Often failure occurs here. (**<u>Note</u>**: <u>not</u> in a pure uniaxial field. This is a common problem with test specimens.)

Example 2

Let's now get a bit more involved and consider the…

Stress Distribution Around a Hole

Figure 8.3 **Configuration of uniaxially loaded plate with a hole**



Large ("infinite") plate subjected to uniform far - field tension

Since the "local specific" of interest is a circle, it makes sense to use *polar* coordinates.

By using the transformations to polar coordinates of Unit #7, we find:

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + V$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} + V$$

$$\sigma_{r\theta} = -\frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial \phi}{\partial \theta}$$

These are, again, defined such that equilibrium equations are automatically satisfied.

where: $f_r = -\frac{\partial V}{\partial r}$ $\quad f_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta}$

and V exists if: $\frac{\partial f_\theta}{\partial r} + \frac{f_\theta}{r} = \frac{1}{r}\frac{\partial f_r}{\partial \theta}$

The governing equation is again:

$$\nabla^4 \phi = -E\alpha\nabla^2(\Delta T) - (1 - \nu)\nabla^2 V$$

for plane stress, isotropic

but the Laplace operator in polar coordinates is:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Let's go through the steps…

<u>Step 1</u>: Assume a $\phi(r, \theta)$

"We know" the correct function is:

$$\phi = \left[A_0 + B_0 \ln r + C_0 r^2 + D_0 r^2 \ln r\right] + \left[A_2 r^2 + \frac{B_2}{r^2} + C_2 r^4 + D_2\right] \cos 2\theta$$

Does this satisfy equilibrium? **<u>It</u> <u>must</u>.**

One can show it satisfies $\nabla^4 \phi$

However, it can also be shown that non-zero values of D_0 result in mulitvalued displacements in the y-direction (v), so we must get $D_0 = 0$

[don't worry about this, just "accept" it…I do!]

Step 2: Determine stresses

Performing the derivatives from the ϕ-σ relations in polar coordinates results in:

$$\sigma_{rr} = \frac{B_0}{r^2} + 2C_0 + \left[-2A_2 - \frac{6B_2}{r^4} - \frac{4D_2}{r^2}\right]\cos 2\theta$$

$$\sigma_{\theta\theta} = -\frac{B_0}{r^2} + 2C_0 + \left[2A_2 + \frac{6B_2}{r^4} + 12C_2 r^2\right]\cos 2\theta$$

$$\sigma_{r\theta} = \left[2A_2 - \frac{6B_2}{r^4} + 6C_2 r^2 - \frac{2D_2}{r^2}\right]\sin 2\theta$$

Note that we have a term involving r^2. As r gets larger, the stresses would become infinite. This is not possible. Thus, the coefficient C_2 must be zero:

$$C_2 = 0$$

So we have five constants remaining:

$$A_2, B_0, B_2, C_0, D_2$$

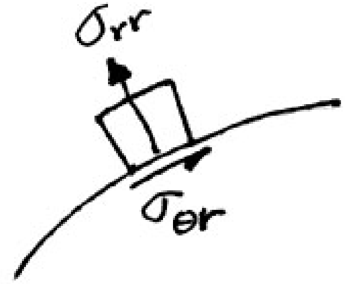
we find these by…

<u>Step 3</u>: Satisfy the boundary conditions

What are the boundary conditions here?

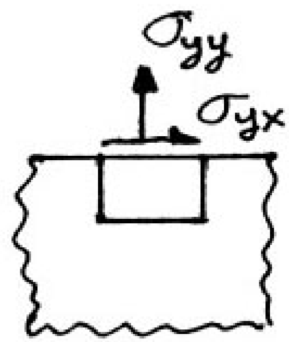
- at the edge of the hole there are no stresses (stress-free edge)

$$\Rightarrow @\ r = a: \quad \sigma_{rr} = 0, \quad \sigma_{\theta r} = 0$$



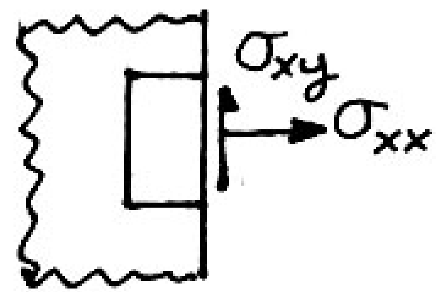
- at the y-edge, there are no stresses

$$\Rightarrow @\ y = \pm\infty: \quad \sigma_{yy} = 0, \quad \sigma_{yx} = 0$$



- at the x-edges, the stress is equal to the applied stress and there is no strain

$$\Rightarrow @\ x = \pm\infty: \quad \sigma_{xx} = \sigma_0, \quad \sigma_{xy} = 0$$



Since we are dealing with polar coordinates, we need to change the last two sets of B. C.'s to polar coordinates. We use:

$$\tilde{\sigma}_{\alpha\rho} = \ell_{\tilde{2}\sigma}\, \ell_{\tilde{\beta}\gamma}\, \ell_{\sigma\gamma}$$

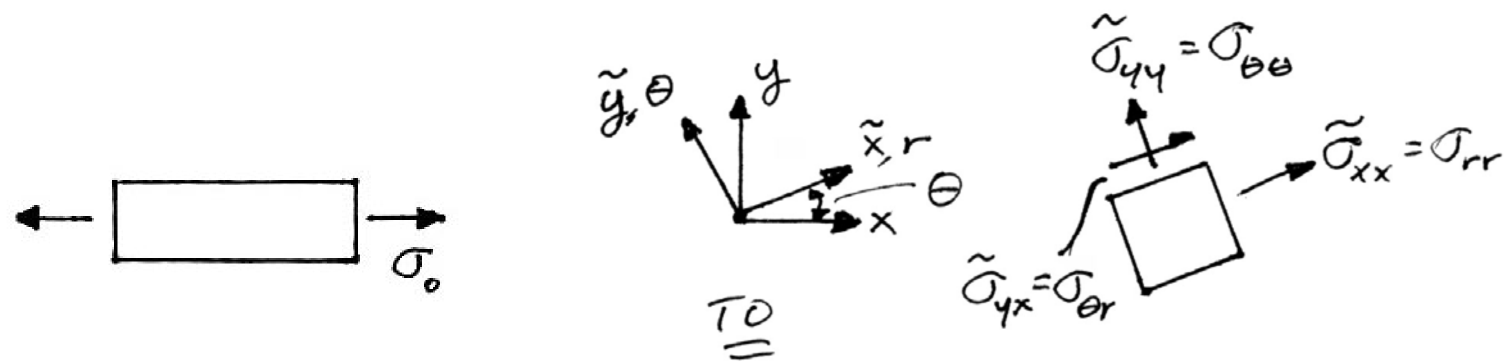
and look at r = ∞

$$\tilde{\sigma}_{xx} = \sigma_{rr} = \cos^2\theta\, \sigma_o$$

$$\tilde{\sigma}_{yy} = \sigma_{\theta\theta} = \sin^2\theta\, \sigma_o$$

$$\tilde{\sigma}_{xy} = \sigma_{r\theta} = -\sin\theta \cos\theta\, \sigma_o$$

Figure 8.4 **Representation of local rotation of stresses from polar to Cartesian system**



We use the double angle trigonometric identities to put this in a more convenient form:

$$\cos^2\theta = \frac{1}{2}\left(1 + \cos 2\theta\right)$$

$$\sin\theta \cos\theta = \frac{1}{2} \sin 2\theta$$

$$\sin^2\theta = \frac{1}{2}\left(1 - \cos 2\theta\right)$$

Thus at r = ∞

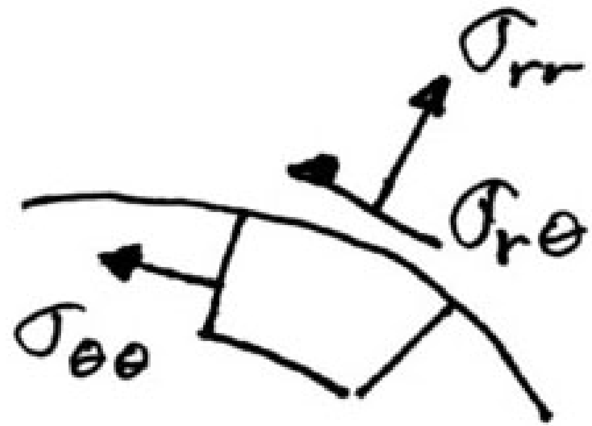
$$\sigma_{rr} = \frac{\sigma_o}{2} + \frac{\sigma_o}{2} \cos 2\theta$$

$$\sigma_{r\theta} = -\frac{\sigma_o}{2} \sin 2\theta$$

Why don't we include $\sigma_{\theta\theta}$?

At the boundary in polar coordinates, this stress does not act on the edge/boundary.

Figure 8.5 **Stress condition at boundary of hole**



So we (summarizing) appear to have 4 B. C.'s:

$$\sigma_{rr} = \frac{\sigma_o}{2} + \frac{\sigma_o}{2} \cos 2\theta \qquad @\ r = \infty$$

$$\sigma_{r\theta} = -\frac{\sigma_o}{2} \sin 2\theta \qquad @\ r = \infty$$

$$\sigma_{rr} = 0 \qquad @\ r = a$$

$$\sigma_{r\theta} = 0 \qquad @\ r = a$$

And we have 5 constants to determine.

What happened?

The condition of σ_{rr} at r = ∞ is really two B. C.'s

– a constant part
– a part multiplying cos 2θ

Going through (and skipping) the math, we end up with:

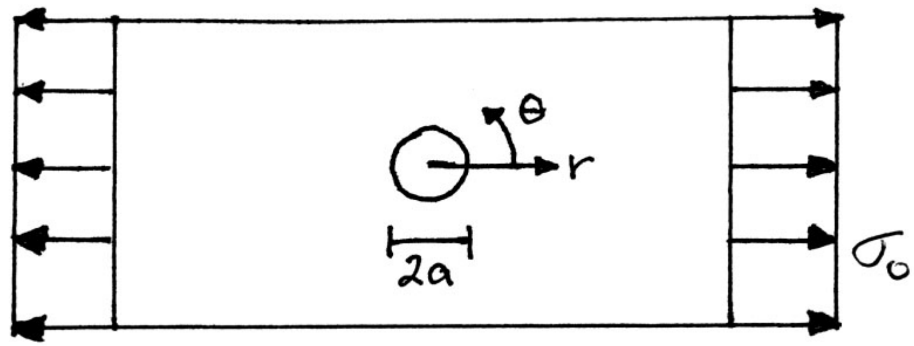
$$\sigma_{rr} = \frac{\sigma_o}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{\sigma_o}{2}\left(1 - 4\frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right)\cos 2\theta$$

$$\sigma_{\theta\theta} = \frac{\sigma_o}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{\sigma_o}{2}\left(1 + 3\frac{a^4}{r^4}\right)\cos 2\theta$$

$$\sigma_{r\theta} = -\frac{\sigma_o}{2}\left(1 + 2\frac{a^2}{r^2} - 3\frac{a^4}{r^4}\right)\sin 2\theta$$

for: …

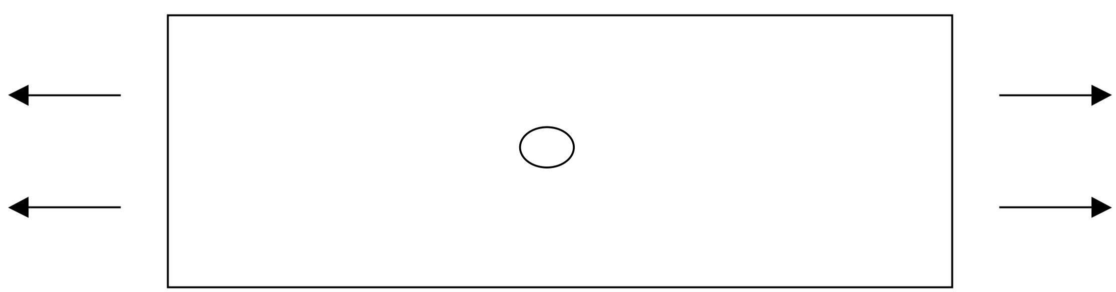
Figure 8.6 **Polar coordinate configuration for uniaxially loaded plate with center hole**



So we have the solution to find the stress field around a hole. Let's consider one important point. Where's the largest stress?

At the edge of the hole. Think of "flow" around the hole:

Figure 8.7 **Representation of stress "flow" around a hole**



@ $\theta = 90°$, $r = a$

$$\sigma_{\theta\theta} = \sigma_{xx} = \frac{\sigma_o}{2}\left(1 + \frac{a^2}{a^2}\right) - \frac{\sigma_o}{2}\left(1 + 3\frac{a^4}{a^4}\right)(-1)$$

$$= 3\,\sigma_o$$

Define the:

<u>Stress</u> <u>Concentration</u> <u>Facto</u>r (SCF) = $\dfrac{\text{local stress}}{\text{far - field stress}}$

SCF = 3 at hole in isotropic plate

The SCF is a more general concept. Generally the "sharper" the discontinuity, the higher the SCF.

The SCF will also depend on the material. For orthotropic materials, it depends on E_x and E_y getting higher as E_x/E_y increases. In a uni-directional composite, can have SCF = 7.

Can do stress functions for orthotropic materials, but need to go to <u>complex</u> <u>variable</u> <u>mapping</u>

--> *(See Lekhnitskii as noted earlier)*

Example 3 -- Beam in Bending

(we'll save for a problem set or a recitation)

There are many other cases (use earlier references)

– circular disks
– rotary disks

.

.

"classic" cases

But…what's really the point? Are these still used?

Yes!

This is a very powerful technique which is especially well-suited for preliminary design and exploratory development

– parametric study
– know assumptions and resulting limitations and then interpret results accordingly
– linear solutions --> can use ***principle of superposition***
– can find many solutions in books

(See attached page)

1. Roark, Raymond J., “Roark’s Formulas for Stresses and Strain, 6th Ed.,” New York, McGraw-Hill, 1989.
2. Peterson, Rudolph E., “Stress Concentration Factors: Charts and Relations Useful in Making Strength Calculations for Machine Parts and Structural Elements,” New York, Wiley, 1974.
3. Pilkey, Walter D., “Formulas for Stress, Strain, and Structural Matrices,” New York, John Wiley, 1994.