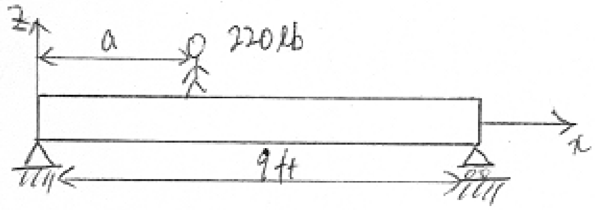
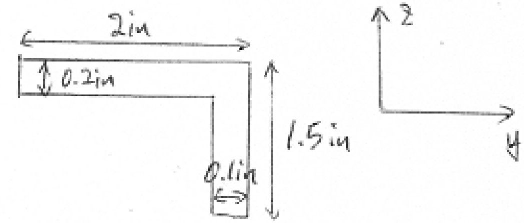
Practice Problems





$E = 30\,Msi$

$\nu = 0.3$

The weight of the man on the beam produces a moment about the y-axis only. The moment about the z-axis is zero. Thus, th expression for the deflections in the y and z directions are

$$\frac{d^2 w}{dx^2} = \frac{M_y I_z}{E(I_y I_z - I_{yz}^2)} \quad \text{———} \quad (1)$$

$$\frac{d^2 v}{dx^2} = \frac{-M_y I_{yz}}{E(I_y I_z - I_{yz}^2)} \quad \text{———} \quad (2)$$

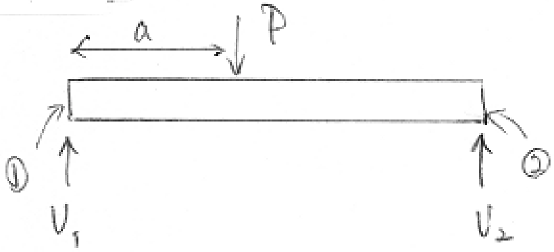
(From unit # 14, p 20)

The expression for the in-plane stress, σ_{xx}, is

$$\sigma_{xx} = \frac{M_y (I_{yz} y - I_z z)}{I_y I_z - I_{yz}^2} \quad \text{———} \quad (3)$$

a) We need to determine the maximum total deflection and location of that deflection. In order to calculate these, we first need to obtain the moment, M_y, along the beam, and then the second moments of inertia.

Moments



$$\Sigma F = 0 \;:\; V_1 + V_2 = P$$

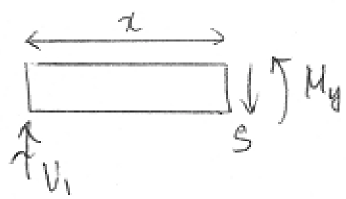
$$\Sigma M_{①} = 0 \;:\; -Pa + V_2 L = 0$$

solving simultaneously, we get

$$V_2 = \frac{Pa}{L}$$

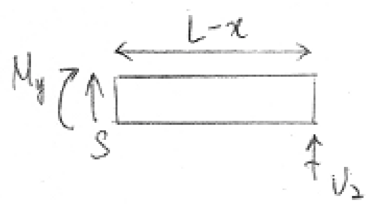
$$V_1 = \frac{P(L-a)}{L}$$

$0 < x < a$:



$$M_y(x) = V_1 x = P\left(1 - \frac{a}{L}\right)x \quad —$$

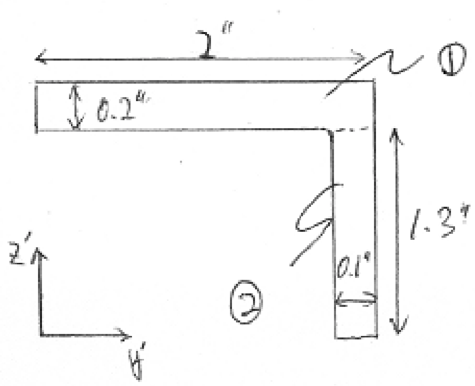
$a < x < L$:



$$M_y(x) = V_2(L-x) = P\frac{a}{L}(L-x) \quad —$$

Moments of Inertia

We need to locate the centroid first.



$$\bar{Y} = \frac{\Sigma A_i \bar{y}_i}{\Sigma A_i} = \frac{0.65}{0.53} = 1.23$$

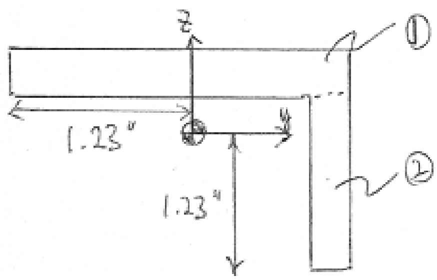
$$\bar{Z} = \frac{\Sigma A_i \bar{z}_i}{\Sigma A_i} = \frac{0.65}{0.53} = 1.23$$

* Values obtained from

section	A_i (in²)	$\bar{z}_i$ (in)	$A_i\bar{z}_i$ (in³)	$\bar{y}_i$ (in)	$A_i\bar{y}_i$ (in³)
1	0.4	1.4	0.56	1	0.4
2	0.13	0.65	0.085	1.95	0.25

$\Sigma A_i = 0.53 \text{ in}^2 \quad \Sigma A_i\bar{z}_i = 0.65 \text{ in}^3 \quad \Sigma A_i\bar{y}_i = 0.65 \text{ in}^3$

Moving the axes to the centroid, we can now calculate the moments of inertia using



$$I_y = \sum_i (I_{yi} + A_i z_i^2)$$

$$I_z = \sum_i (I_{zi} + A_i y_i^2)$$

$$I_{yz} = \sum_i (I_{yzi} + A_i y_i z_i)$$

I_y :

Section	b_i (in)	h_i (in)	I_{yi} (in⁴)	z_i (in)	A_i (in²)	$A_i z_i^2$ (in
1	2	0.1	1.67×10^{-4}	0.17	0.4	0.0116
2	0.1	1.3	0.0183	-0.58	0.13	0.0437
Σ	–	–	0.0185	–	–	0.0553

$\therefore \quad I_y = 0.0185 + 0.0553$

$\Rightarrow \quad I_y = 0.074 \text{ in}^4$

I_z:

Section	b_i (in)	h_i (in)	I_z (in^4)	y_i (in)	A_i (in^2)	$A_i y_i^2$ (in^4)
1	0.1	2	0.067	-0.23	0.4	0.0212
2	1.3	0.1	1.08×10^{-4}	0.72	0.13	0.0674
Σ	—	—	0.067	—	—	0.0886

$$\therefore \quad I_z = 0.067 + 0.0886$$

$$\Rightarrow \quad I_z = 0.156 \text{ in}^4$$

I_{yz}:

Section	y_i (in)	z_i (in)	A_i (in^2)	$y_i z_i A_i$ (in^4)
1	-0.23	0.17	0.4	-0.0156
2	0.72	-0.58	0.13	-0.0543
Σ	—	—	—	-0.07

$$\therefore \quad I_{yz} = -0.07 \text{ in}^4$$

Rewriting,

$$I_y = 0.074 \text{ in}^4$$

$$I_z = 0.156 \text{ in}^4$$

$$I_{yz} = -0.07 \text{ in}^4$$

Now, let's find the maximum deflection and location using equation 1 and 2. Note that equations 1 and 2 show that the y and z

derivatives are both proportional to M_y, and they also have the sam boundary conditions. Thus, the maximum deflection location for both directions will be identical. Moreover, if we let the coefficients of M_y in equations ① and ② be

$$\frac{1}{I_w} = \frac{I_z}{I_y I_z - I_{yz}^2} \qquad \text{⑥}$$

$$\frac{1}{I_v} = \frac{I_{yz}}{I_y I_z - I_{yz}^2} \qquad \text{⑦}$$

then equations ① and ② become

$$\frac{d^2 w}{dx^2} = \frac{M_y}{E I_w} \qquad \text{⑧}$$

$$\frac{d^2 v}{dx^2} = -\frac{M_y}{E I_v} \qquad \text{⑨}$$

Since the form of the differential equations in ⑧ and ⑨ are identical and the boundary conditions are identical, the form of the solution are also identical. Thus, only one of the two equations need to b solved. Let's consider equation ⑧.

$0 < x < a$:

$$\frac{d^2 w}{dx^2} = \frac{M_y}{E I_w} = \frac{1}{E I_w} P\left(1 - \frac{a}{L}\right)x \qquad (\text{using eq. ④});$$

$$\Rightarrow \frac{dw}{dx} = \frac{P}{E I_w}\left(1 - \frac{a}{L}\right)\frac{x^2}{2} + C_1 \qquad \text{⑩}$$

$$\Rightarrow \quad w = \frac{P}{EI_w}\left(1-\frac{a}{L}\right)\frac{x^3}{6} + C_1 x + C_2 \quad \text{——— (11}$$

B.C.'s @ $x=0$: $w=0 \Rightarrow C_2 = 0$

@ $x=a$: $\frac{dw}{dx}$ matches with region $a<x<L$ ——— (

w matches with region $a<x<L$ ——— (1

$a<x<L$:

$$\frac{d^2w}{dx^2} = \frac{M}{EI_w} = \frac{Pa}{EI_w L}(L-x) \qquad \text{(using eq. ⑤)}$$

$$\Rightarrow \frac{dw}{dx} = \frac{Pa}{EI_w L}\left(Lx - \frac{x^2}{2}\right) + C_3 \quad \text{——— (14)}$$

$$\Rightarrow \quad w = \frac{Pa}{EI_w L}\left(L\frac{x^2}{2} - \frac{x^3}{6}\right) + C_3 x + C_4 \quad \text{——— (15}$$

B.C.'s @ $x = L$: $w=0$

$$\Rightarrow \frac{PaL^2}{2EI_w} - \frac{PaL^2}{6EI_w} + C_3 L + C_4 = 0 \text{——— (16}$$

@ $x=a$: same as B.C.'s in (12) and (13).

The B.C.'s in (12) and (13) can be expressed as

$$\frac{dw}{dx} \text{ @ } x=a : \quad \frac{P}{EI_w}\left(1-\frac{a}{L}\right)\frac{a^2}{2} + C_1 = \frac{Pa}{EI_w L}\left(La - \frac{a^2}{2}\right) + C_3$$

$$\Rightarrow \frac{a^2}{2} - \frac{a^3}{2L} + C_1\frac{EI_w}{P} = a^2 - \frac{a^3}{2L} + C_3\frac{EI_w}{P}$$

——— (1

$$w \text{ @ } x=a : \quad \frac{P}{EI_w}\left(1-\frac{a}{L}\right)\frac{a^3}{6} + C_1 a = \frac{Pa}{EI_w L}\left(L\frac{a^2}{2} + \frac{a^3}{6}\right) + C_3 a +$$

$$\Rightarrow \frac{a^3}{6} - \frac{a^4}{6L} + C_1 a\frac{EI_w}{P} = \frac{a^3}{2} + \frac{a^4}{6L} + C_3 a\frac{EI_w}{P} + C$$

Our unknowns are C_1, C_3 and C_4, and we have three equatio(ns) ⑯, ⑰ and ⑱. Thus, we can find the three unknowns.

Solving for the unknowns, we get,

$$C_4 = \frac{P}{EI_w}\frac{a^3}{6}$$

$$C_3 = \frac{P}{EI_w}\left(-\frac{a^3}{6L} - \frac{aL}{3}\right)$$

$$C_1 = \frac{P}{EI_w}\left(\frac{a^2}{2} - \frac{a^3}{6L} - \frac{aL}{3}\right)$$

So, the deflections and slopes are

$0 < x < a$: $$w(x) = \frac{P}{EI_w}\left[\left(1-\frac{a}{L}\right)\frac{x^3}{6} + \left(\frac{a^2}{2} - \frac{a^3}{6L} - \frac{aL}{3}\right)x\right]$$

$a < x < L$: $$w(x) = \frac{P}{EI_w}\left[\frac{ax^2}{2} - \frac{ax^3}{6L} + \left(-\frac{a^3}{6L} - \frac{aL}{3}\right)x + \left(\frac{a^3}{6}\right)\right]$$

$0 < x < a$: $$\frac{dw(x)}{dx} = \frac{P}{EI_w}\left[\left(1-\frac{a}{L}\right)\frac{x^2}{2} + \left(\frac{a^2}{2} - \frac{a^3}{6L} - \frac{aL}{3}\right)\right]$$

$$\frac{dw(x)}{dx} = \frac{P}{EI_w}\left[ax - \frac{ax^2}{2L} - \frac{a^3}{6L} - \frac{aL}{3}\right]$$

The maximum $w(x)$ occurs when

$$\frac{dw}{dx} = 0 \quad \text{and} \quad \frac{dw}{da} = 0 \qquad \text{———} \quad ⑲$$

For $0 < x < a$, equation ⑲ becomes

$$\frac{dw}{dx} = \left(1-\frac{a}{L}\right)\frac{x^2}{2} + \left(\frac{a^2}{2} - \frac{a^3}{6L} - \frac{aL}{3}\right) = 0 \qquad \text{———} \quad ⑳$$

$$\Rightarrow \quad \frac{x^2}{2} = -\left(\frac{L}{L-a}\right)\left(\frac{a^2}{2} - \frac{a^3}{6L} - \frac{aL}{3}\right) \qquad \text{——— } (21)$$

$$\frac{dw}{da} = -\frac{x^3}{6L} + \left(a - \frac{a^2}{2L} - \frac{L}{3}\right)x = 0 \qquad \text{——— } (22)$$

Substituting equation (21) into equation (22), we get,

$$-\left(\frac{x^2}{2}\right)\left(\frac{x}{3L}\right) + \left(a - \frac{a^2}{2L} - \frac{L}{3}\right)x = 0$$

$$\Rightarrow -\frac{x}{3(L-a)}\left(\frac{a^2}{2} - \frac{a^3}{6L} - \frac{aL}{3}\right) + \left(a - \frac{a^2}{2L} - \frac{L}{3}\right)x = 0$$

$$\Rightarrow x\left[-\frac{1}{3(L-a)}\left(\frac{a^2}{2} - \frac{a^3}{6L} - \frac{aL}{3}\right) + \left(a - \frac{a^2}{2L} - \frac{L}{3}\right)\right] = 0 \quad \text{—}$$

The solution to equation (23) is $x=0$, which is trivial, or

$$-\frac{1}{3(L-a)}\left(\frac{a^2}{2} - \frac{a^3}{6L} - \frac{aL}{3}\right) + \left(a - \frac{a^2}{2L} - \frac{L}{3}\right) = 0$$

$$\Rightarrow \frac{4}{9}\left(\frac{a}{L}\right)^3 - \frac{4}{3}\left(\frac{a}{L}\right)^2 + \frac{11}{9}\left(\frac{a}{L}\right) - \frac{1}{3} = 0 \quad \text{——}$$

The solutions equation (24) are

$$\frac{a}{L} = 1 \text{ or } \frac{1}{2} \text{ or } \frac{3}{2}$$

Since $a=L$ and $a=\frac{3}{2}L$ are not the solution within our boun

a must equal to $\frac{L}{2}$. This result can also be obtained from

symmetry arguments. Thus, plugging $a=\frac{L}{2}$ into equation (20), we g

$$\frac{dw}{dx} = \frac{x^2}{4} + \left(\frac{L^2}{8} - \frac{L^2}{6\cdot 8} - \frac{L^2}{6}\right) = 0$$

$$\Rightarrow \quad x^2 = \frac{L^2}{4}$$

$$\therefore \quad x = \pm\frac{L}{2}$$

Since x cannot be $-\frac{L}{2}$, it must be $\frac{L}{2}$. Therefore, the maxim occurs when $a = \frac{L}{2}$ and at $x = \frac{L}{2}$. The deflection at $x = \frac{L}{2}$ can be found by plugging $a = \frac{L}{2}$ into the equation for $0 < x < a$.

$$w\left(\frac{L}{2}\right) = \frac{P}{EI_w}\left[\frac{1}{2}\frac{1}{6}\frac{L^3}{8} + \left(-\frac{L^2}{16}\right)\frac{L}{2}\right]$$

$$\Rightarrow \quad w\left(\frac{L}{2}\right) = \frac{P}{EI_w}\left(-\frac{L^3}{48}\right) \qquad \text{——— (25)}$$

Similarly, the maximum y deflection is at $x = \frac{L}{2}$ when $a = \frac{L}{2}$.

$$v\left(\frac{L}{2}\right) = \frac{P}{EI_v}\left(+\frac{L^3}{48}\right) \qquad \text{——— (26)}$$

Since

$$I_w = \frac{I_y I_z - I_{yz}^2}{I_z} = \frac{(0.074)(0.156) - (-0.07)^2}{0.156}$$

$$\Rightarrow \quad I_w = 0.0426 \text{ in}^4$$

$$I_v = \frac{I_y I_z - I_{yz}^2}{I_{yz}} = \frac{(0.074)(0.156) - (-0.07)^2}{-0.07}$$

$$\Rightarrow \quad I_v = -0.0949 \text{ in}^4$$

Therefore, the values for $w(\frac{L}{2})$ and $v(\frac{L}{2})$ are:

$$w(\tfrac{L}{2}) = \frac{220\,lb}{(30Msi)(0.0426in^4)}\left(-\frac{(9\times12)^3}{48}\right)$$

$$\therefore \boxed{w(\tfrac{L}{2}) = -4.52 \text{ in}}$$

$$v(\tfrac{L}{2}) = \frac{220\,lb}{(30Msi)(0.0949\,in^4)}\left(+\frac{(9\times12)^3}{48}\right)$$

$$\therefore \boxed{v(\tfrac{L}{2}) = -2.03 \text{ in}}$$

b) From equation ③, the stress, σ_{xx} is

$$\sigma_{xx} = \frac{M_y(I_{yz}y - I_z z)}{I_y I_z - I_{yz}^2} = \frac{(-0.07y - 0.156z)}{(0.074)(0.156)-(-0.07)^2} M_y.$$

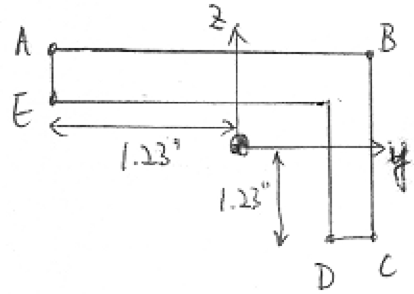
For $0<x<a$, M_y is maximum at $x=\frac{L}{2}$ when $a=\frac{L}{2}$ (see equatio

Thus, σ_{xx} is maximum at

$$\sigma_{xx} = \frac{(-0.07y-0.156z)}{0.0066}\frac{PL}{4} = (220\,lb)(9\times12)$$

$$\Rightarrow \sigma_{xx} = -6.3\times10^4 y + 14\times10^4 z \quad (psi)$$

σ_{xx} is larger at points that are further away from the ne axis.



Point	y_i (in)	z_i (in)	σ_{xx} (psi)
A	-1.23	0.27	11.5×10^4
B	0.77	0.27	-1.1×10^4
C	0.77	-1.23	-22×10^4
D	0.67	-1.23	-21×10^4
E	-1.23	0.07	8.7×10^4

Thus, from the table, the maximum value is

$$\boxed{\sigma_{xx\,max} = -22 \times 10^4 \text{ psi}}$$ @ point C.