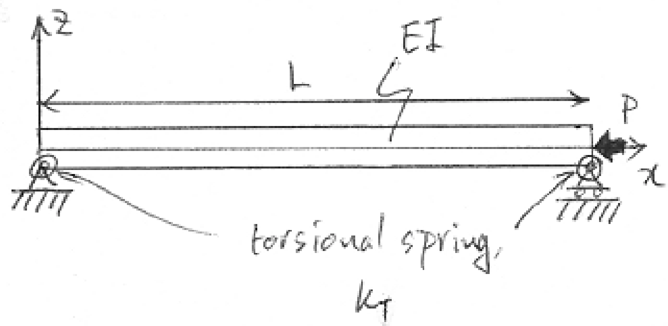
Solutions to Home Assignment #10

Warm-up Exercises



Governing differential equation for a column (unit #16, p9)

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) - \frac{d}{dx}\left(F\frac{dw}{dx}\right) = \quad \text{—————}$$

1. The general solution to equation ① is

$$w(x) = A\sin\lambda x + B\cos\lambda x + Cx + D \quad \text{—————} \; ②$$

where $\lambda = \sqrt{\frac{P}{EI}}$

The boundary conditions need to be found to solve for the constan A, B, C, D and P_{cr}. At both ends the displacment is zero due to the pin supports. Thus,

$$w(0) = w(L) = 0 \quad \text{—————} \; ③$$

Also, due to the torsional springs, the moment at both ends are equal to the torsional spring constant k_T times the angle of rotation $\theta = \frac{dw}{dx}$. Thus,

* Note : B.C.'s in equation ④ are formally derived from $\Sigma M = 0$.

$$\left.\begin{aligned} M(0) &= EI\frac{d^2w(0)}{dx^2} = k_T\frac{dw(0)}{dx} \\ M(L) &= EI\frac{d^2w(L)}{dx^2} = -k_T\frac{dw(L)}{dx} \end{aligned}\right\} \quad \text{—————} \; ④$$

Applying the boundary conditions at $x=0$, we get

$$③ \longrightarrow \quad w(0) = A\sin(0) + B\cos(0) + C\cdot 0 + D = 0$$

$$\Rightarrow \quad w(0) = B + D = 0 \quad \text{———} \quad ⑤$$

$$④ \longrightarrow \quad EI\left(-\lambda^2 A\sin\lambda\cdot 0 - \lambda^2 B\cos\lambda\cdot 0\right)$$

$$= k_T\left(\lambda A\cos\lambda 0 - \lambda B\sin\lambda\cdot 0 + C\right)$$

$$\Rightarrow \quad -EI\lambda^2 B = k_T\lambda A + k_T C$$

$$\Rightarrow \quad C = -\lambda A - EI\lambda^2 \underbrace{B}_{= -D \text{ (from ⑤)}} \frac{1}{k_T} \quad \text{———} \quad ⑥$$

Thus, we get

$$w(x) = A\sin\lambda x - D\cos\lambda x - \left(\lambda A - \frac{EI}{k_T}\lambda^2 D\right)x + D$$

$$\Rightarrow w(x) = A(\sin\lambda x - \lambda x) + D\left(1 + \frac{EI}{k_T}\lambda^2 x - \cos\lambda x\right) \quad \text{—}$$

Next, we apply the boundary conditions at $x=L$.

$$③ \longrightarrow \quad w(L) = A(\sin\lambda L - \lambda) + D\left(1 + \frac{EI}{k_T}\lambda^2 L - \cos\lambda L\right) = 0 \quad -$$

$$④ \longrightarrow \quad EI\left[A(-\lambda^2\sin\lambda L) + D(\lambda^2\cos\lambda L)\right]$$

$$= -k_T\left[A(\lambda\cos\lambda L - \lambda) + D\left(\frac{EI}{k_T}\lambda^2 + \lambda\sin\lambda L\right.\right.$$

$$\Rightarrow A\left[k_T(\lambda\cos\lambda L - \lambda) - EI\lambda^2\sin\lambda L\right] + D\left[k_T\left(\frac{EI\lambda^2}{k_T} + \lambda\sin\lambda L\right) + EI(\lambda^2\cos\right.$$

$$\Rightarrow A\left[\frac{k_T L^2}{EI}(\lambda\cos\lambda L-\lambda)-\lambda^2L^2\sin\lambda L\right]+D\left[\lambda^2L^2+\frac{k_T L^2}{EI}\lambda\sin\lambda L+\lambda^2L^2\cos\lambda L\right. \qquad —(9$$

Setting $\bar{\lambda}=\lambda L$ and $\bar{k}=\frac{k_T L}{EI}$, and expressing equations (8) and (
in matrix form, we get

$$\begin{bmatrix} \bar{k}(\sin\bar{\lambda}-\bar{\lambda}) & \bar{\lambda}^2+\bar{k}(1-\cos\bar{\lambda}) \\ \bar{k}(\bar{\lambda}\cos\bar{\lambda}-\bar{\lambda})-\bar{\lambda}^2\sin\bar{\lambda} & \bar{k}\bar{\lambda}\sin\bar{\lambda}+\bar{\lambda}^2(1+\cos\bar{\lambda}) \end{bmatrix}\begin{Bmatrix} A \\ D \end{Bmatrix}=\Big\{ \qquad —($$

To obtain non-trivial solutions, we set the determinant of the m
to zero.

$$\bar{k}(\sin\bar{\lambda}-\bar{\lambda})\left(\bar{k}\bar{\lambda}\sin\bar{\lambda}+\bar{\lambda}^2(1+\cos\bar{\lambda})\right)$$
$$-\left[\bar{k}(\bar{\lambda}\cos\bar{\lambda}-\bar{\lambda})-\bar{\lambda}^2\sin\bar{\lambda}\right]\left[\bar{\lambda}^2+\bar{k}(1-\cos\bar{\lambda})\right]=($$

$$\Rightarrow \bar{k}\left(\bar{k}\bar{\lambda}\sin^2\bar{\lambda}+\bar{\lambda}^2\sin\bar{\lambda}+\cancel{\bar{\lambda}^2\sin\bar{\lambda}\cos\bar{\lambda}}-\bar{k}\bar{\lambda}^2\sin\bar{\lambda}-\cancel{\bar{\lambda}^3}-\bar{\lambda}^3\cos\bar{\lambda}\right)$$
$$-\left(\bar{k}\bar{\lambda}^3\cos\bar{\lambda}-\cancel{\bar{k}\bar{\lambda}^3}-\bar{\lambda}^4\sin\bar{\lambda}+\bar{k}^2\bar{\lambda}\cos\bar{\lambda}-\bar{k}^2\bar{\lambda}-\bar{k}\bar{\lambda}^2\sin\bar{\lambda}\right.$$
$$\left.-\bar{k}^2\bar{\lambda}\cos^2\bar{\lambda}+\bar{k}^2\bar{\lambda}\cos\bar{\lambda}+\cancel{\bar{k}\bar{\lambda}^2\sin\bar{\lambda}\cos\bar{\lambda}}\right.$$

$$\Rightarrow \bar{k}^2\bar{\lambda}(\sin^2\bar{\lambda}+\cos^2\bar{\lambda}+1)+2\bar{k}\bar{\lambda}^2\sin\bar{\lambda}-\bar{k}^2\bar{\lambda}^2\sin\bar{\lambda}-2\bar{\lambda}^3\cos\bar{\lambda}$$
$$+\bar{\lambda}^4\sin\bar{\lambda}-2\bar{k}^2\bar{\lambda}\cos\bar{\lambda}$$

$$\Rightarrow \boxed{\bar{\lambda}\left\{\bar{k}^2(2-2\cos\bar{\lambda}-\bar{\lambda}\sin\bar{\lambda})+2\bar{k}\bar{\lambda}^2(\sin\bar{\lambda}-\bar{\lambda}\cos\bar{\lambda})+\bar{\lambda}^3\sin\bar{\lambda}\right\}=0} \qquad —(11$$

2. If $k_T \to 0$, we get from equation ⑪

$$\sin \bar{\lambda} = 0$$

$$\Rightarrow \lambda L = n\pi$$

$$\Rightarrow \frac{P}{EI} L^2 = n^2 \pi \qquad \left(\lambda = \sqrt{\frac{P}{EI}} \right)$$

$$\therefore P = \frac{n^2 \pi^2 EI}{L^2}$$

$$\Rightarrow \boxed{P_{critical} = \frac{\pi^2 EI}{L^2}}$$

This critical load is the same as that of a simply-supported colu[mn]. This is expected because if $k_T \to 0$, which means that there [is] no effective torsional spring constant, the problem simply become[s] a simply-supported column.

3. If $k_T \to \infty$, we get from equation ⑪

$$2 - 2\cos\bar{\lambda} - \bar{\lambda}\sin\bar{\lambda} = 0$$

$$\Rightarrow \bar{\lambda} = 2n\pi$$

$$\therefore P = \frac{4n^2\pi^2 EI}{L^2}$$

$$\Rightarrow \boxed{P_{critical} = \frac{4\pi^2 EI}{L^2}}$$

The critical load is same as that of a clamped-clamped column. If the torsional spring constant is infinitely stiff, the spring will not allow any rotation at the ends. Therefore, this condit should be equivalent to the clamped-clamped case which does no allow the slope to be finite (i.e. $\frac{dw}{dx} = 0$)