- Dimen Analysis
  - π Theorem
  + Examples
- Dominant balance on scaling arg
- Rayleigh problem.

Lecture 6 — Sept 15

3·1 A) Dimensional Analysis — π Theorem.

B) Dominant balance and viscous flow classification

Reading

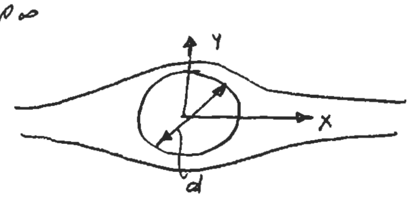
White : 81 – 88, 93, 104 – 107, 114 – 119, 132 – 141

Sch : 13 – 18.

A) π Theorem.

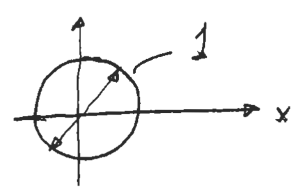
Process of changing from standard (m, kg, s) to natural units is called non-dimensionalization

Example : flow past a cylinder



Standard units : $u(x, y, U_\infty, p_\infty, d)$
$p(x, y, U_\infty, p_\infty, d)$

Natural Units : $U(X, Y)$, $P(X, Y)$



$P = p / \rho_\infty U_\infty^2$, $R = \rho / \rho_\infty = 1$

Number of independant variables has been reduced from 5 → 2

Π Theorem: "Given a problem with "s" scales or independant variables, in "u" standard units, there are $\underline{\underline{p = s - u}}$ non-dimensional variables sufficient to define the problem"

Ex 1 Inviscid, incomp. flow over cylinder

indep. vars (scales)	Units
x	L
y	L
d	L
U_∞	L/T
ρ_∞	M/L^3
$s = 5$	$u = 3$

$\longrightarrow p = 2$

$\Rightarrow U(X, Y)$
$P(X, Y)$

Π group

$U = (u/U_\infty)$

$X = (x/d)$

$P = (p/\rho_\infty U_\infty^2)$

Ex 2 Viscous, incomp flow

$+ \ \nu_\infty$	M/LT
$s = 6$	$u = 3$

$\longrightarrow p = 3$

$\Rightarrow U(X, Y; Re)$

non-dim variable

non-dim. parameter

Ex 3 Compressible Viscous Flow

$$\underbrace{+\ a_\infty,\ c_p,\ k_\infty,\ \theta_\infty}_{s=10} \qquad \underbrace{L/T,\ L^2/T^2K,\ \frac{ML}{T^2K},\ \overset{\text{temp}}{K}}_{u=4}$$

$$p = 10 - 4 = 6.$$

$$U(X, Y, Re, Ma, Pr, \gamma)$$

Ex $\dfrac{V_\infty d}{\nu_\infty}$, $\dfrac{u_\infty}{a_\infty}$, $\dfrac{c_p \mu_\infty}{k_\infty}$, $\gamma = c_p/c_v = \dfrac{c_p}{c_p - R}$

Deriving Non-Dimensional Groups: Use viscous incomp flow as Ex.

scales $\rightarrow d^{\alpha_1} \cdot U_\infty^{\alpha_2} \cdot \rho_\infty^{\alpha_3} \cdot \mu_\infty^{\alpha_4} = M^0 \cdot L^0 \cdot T^0$ = non dimensional quantity

$$= L^{\alpha_1} \cdot (L/T)^{\alpha_2} \cdot (M/L^3)^{\alpha_3} \cdot (M/LT)^{\alpha_4}$$

exponents must vanish

$$\alpha_1 + \alpha_2 - 3\alpha_3 - \alpha_4 = 0$$
$$-\alpha_2 - \alpha_4 = 0 \qquad \text{(rank 3 matrix)}$$
$$\alpha_3 + \alpha_4 = 0$$

Choose $\alpha_1 = 1 \Rightarrow \alpha_2 = 1,\ \alpha_3 = 1,\ \alpha_4 = -1$

$$\Rightarrow \frac{d U \rho_\infty}{\mu_\infty} \rightarrow Re \ //$$

Add x & y.

$$x^{\alpha_1} \cdot y^{\alpha_2} \cdot d^{\alpha_3}\ U_\infty^{\alpha_4}\ \rho_\infty^{\alpha_5}\ \mu_\infty^{\alpha_6} = \text{non-dim quantity}$$

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 3\alpha_5 - \alpha_6 = 0$$
$$-\alpha_4 - \alpha_6 = 0$$
$$\alpha_5 + \alpha_6 = 0$$

Select 3 arbitrary $\rightarrow \alpha_1 = \alpha_2 = \alpha_4 = 1 \Rightarrow \alpha_6 = -1,\ \alpha_5 = 1,\ \alpha_3 = -1$

$$x\ y \left(\frac{1}{d}\right) \cdot \frac{U_\infty \rho_\infty d}{\mu_\infty\ d} \Rightarrow \left(\frac{x}{d}\right), (y/d),\ Re. \ //$$

Dominant Balance and viscous flow classification

incomp ...

$$\frac{D\vec{u}}{Dt} = -\frac{\nabla p}{\rho} + \nu \nabla^2 u$$

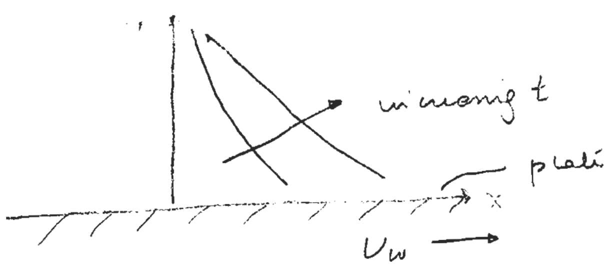
1 - 2   3   4

In general, no exact solution is possible. Depending on which terms balance $\nu \nabla^2 u$, we can have solutions for special types of flows or geometries

Examples

1. Impulsively started flow (Rayleigh, Stokes 1st problem) 1 - 4
2. Pressure-driven steady duct flows (3-4) (Poiseuille)

Note - when $V \cdot \nabla$ can be neglected → linear solutions



$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

B.Cs: $t \leq 0$: $u = 0$ for all y

$t > 0$: $u = U_w$ at $y = 0$

$u = 0$ $y \to \infty$.

$$\frac{\partial v}{\partial y} = 0, \quad V = 0 \quad \text{(parallel flow)}$$

$$\frac{\partial u}{\partial x} = 0$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \qquad \text{(identical to heat conduction)}$$

Scales	Units
U_w	L/T
ν	L^2/T

$t_{ref} = \nu / U_w^2$

$L_{ref} = \nu / U_w$

$$\rightarrow \quad \frac{\partial u^*}{\partial t^*} = \frac{\partial^2 u^*}{\partial y^{*2}} \qquad \nu^* = 1$$

Ordering

$$\frac{\partial u}{\partial(\nu t)} = \frac{\partial^2 u}{\partial y^2}$$

Consider case where $t \to \infty$, $\nu \to 0$, we must have

$$(\nu t) = O(y^2)$$

or $\quad y/\sqrt{\nu t} = O(1)$ for any ν, t, y

Suggests transformation

$$\eta = y/\sqrt{\nu t}$$

$$u = U_w f(\eta)$$

$$\tfrac{1}{2}\eta \frac{\partial u}{\partial \eta} + \frac{\partial^2 u}{\partial \eta^2} = 0 \quad \text{or} \quad \tfrac{1}{2}\eta f' + f'' = 0$$

ODE for $f(\eta)$

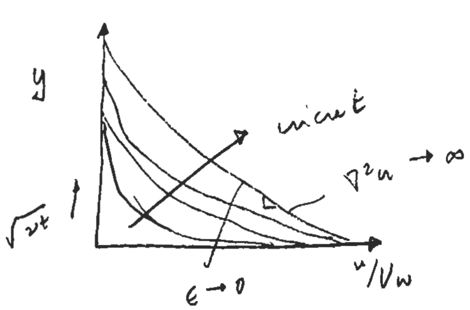
B.C.: $f = 1$ at $\eta = 0$

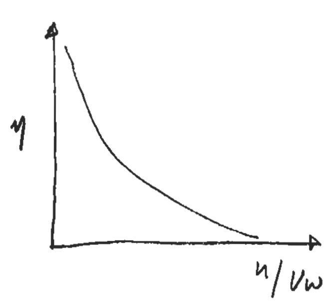
$f = 0$ at $\eta = \infty$

Solution is

$$u(\eta) = U_w \,\mathrm{erfc}(\eta/2)$$

where $\mathrm{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2}\, dx$

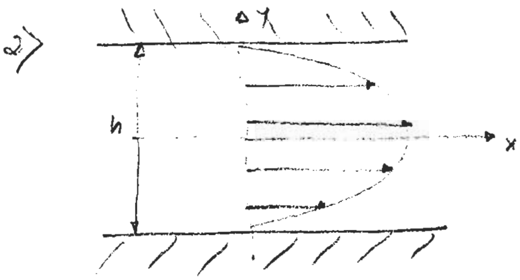




$\sqrt{\nu t} = O(\delta)$

$\uparrow$ b.l. thickness

2)



Steady flow between 2 plates with a constant pressure gradient

$v = 0$, $u = 0$ at $y = \pm h/2$

$\frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0$ from continuity

(parallel flow)

$$\frac{dp}{dx} = \mu \frac{d^2 u}{dy^2}$$

$$\Rightarrow u = -\frac{1}{2\mu}\left(\frac{dp}{dx}\right)\left(\left(\frac{h}{2}\right)^2 - y^2\right) \qquad \left(\delta \sim \sqrt{\nu}\right)$$

low $Re \ll 1$

$$x^* = x/L_{ref} \quad v^* = \frac{v_{ref}}{L_{ref}} \quad t^* = t/_{L_{ref}/U_{ref}} \quad p^* = p/_{\frac{\mu_{ref} U_{ref}}{L_{ref}}}$$

$$\Rightarrow Re \frac{Du^*}{Dt} = -\nabla^* p^* + \nabla^{*2} v^*$$

$\therefore$ $Re \ll 1$ we can neglect inertia

$$-\frac{\nabla p}{\rho} + \nu \nabla^2 u = 0$$

Note there is no restriction on Re in duct flow.

For $Re \gg 1$ all terms are important (1 – 4). We can simplify $\nu \nabla^2 u$, it cannot be dropped near a solid boundary with no-slip conditions

$$\left(\frac{1}{Re}\right)(\nabla^2 u) \sim O(1)$$

$\nabla^2 u \sim Re$ near a wall.

$\hookrightarrow$ asymptotic expansion $1/Re$ as parameter

Another approach

$\nu \nabla^2 u$ balances ∇p near the wall.

$\downarrow$

$\sim \frac{\partial p}{\partial x}$