7.3) Equilibrium Flows. Boundary Layers. (Goal is to develop closure relations for IBL

Outer wake profile $\frac{u_e - u}{u^*} = f(y/\delta, \beta)$

Clauser (1954, 1956) developed the idea of equilibrium flows.

$$\beta = \frac{\delta^*}{\tau_w}\frac{dp_e}{dx} = \text{const}$$

↑ Clauser's parameter

β = constant flows correspond to a power-law free stream $u_e \sim c x^m$ analogous to F-S for laminar flows. Careful experiment to show eq. flow

Clauser defined a defect thickness

$$\Delta = \int \frac{u_e - u}{u^*}\,dy = \int \left(1 - \frac{u}{u_e}\right) dy \cdot \frac{u_e}{u^*} = \delta^* \sqrt{\frac{2}{c_f}}$$

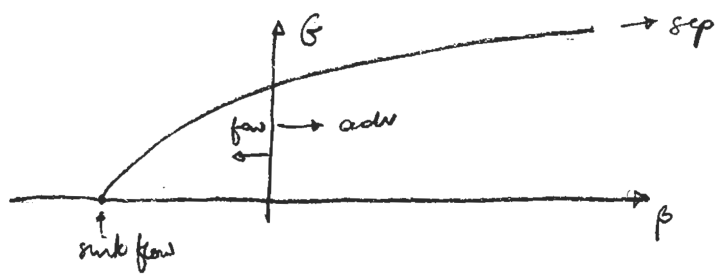
Define analogous shape parameter for defect profile

$$G = \frac{\int \left(\frac{u_e - u}{u^*}\right)^2 dy}{\int \left(\frac{u_e - u}{u^*}\right) dy} = \frac{u_e}{u^*}\,\frac{\delta^* - \theta}{\delta^*} = \sqrt{\frac{2}{c_f}}\left(\frac{H-1}{H}\right)$$

Note if G = const., H varies in x since c_f changes with x

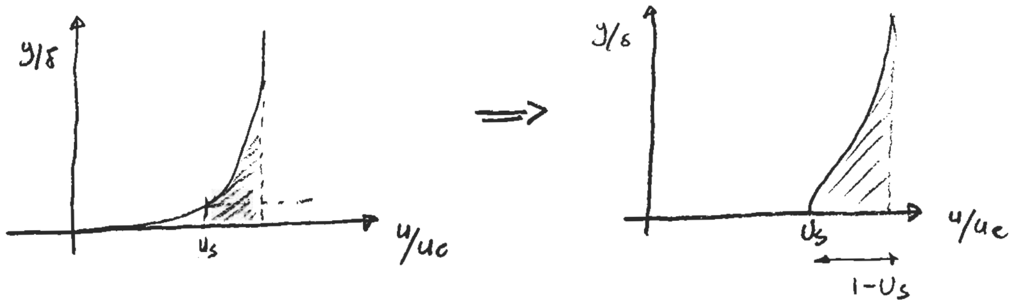
Implication of equilibrium flow: G is const in x, β const.

$$G = G(\beta)$$



Empirical fit $\rightarrow$ $G^2 = A^2(1 + B\beta)$ — linear in β.

Analyse why G^2/Δ^2 is linear in β. Consider outer wake profile



$U_s \equiv$ slip velocity

We can write Coles outer profile as (assumed wake profile)

$$\frac{u}{U_e} = U_s + (1-U_s)\frac{1}{2}\left[1 - \cos(\pi y/\delta)\right]$$

We can calculate integral thicknesses.

$$\frac{\delta^*}{\delta} = \frac{1-U_s}{2}$$

$$\frac{\theta}{\delta} = \frac{\delta^*}{\delta} - \frac{3}{8}(1-U_s)^2$$

$$\therefore \quad \frac{H-1}{H} = \frac{3}{4}(1-U_s)$$

Recall that the eddy viscosity is given by

$$\nu_t = \frac{-\overline{u'v'}}{\frac{\partial u}{\partial y}}$$

or $$-\rho\overline{u'v'} = \mu_t \frac{\partial \bar{u}}{\partial y}$$

Postulate a model for eddy viscosity in outer layer

Karman & Prandtl

$$l = \text{const.} \qquad (\underset{\substack{\uparrow \\ \approx 0.09}}{K}\overset{\text{outer length scale}}{\delta})$$

$$\mu_t = \rho K^2 \delta^2 \left|\frac{\partial u}{\partial y}\right|$$

Clauser

$$\mu_t = \underset{\substack{\uparrow \\ 0.014 - \\ \approx 0.016}}{K}\rho u_e \delta^* \qquad \nu_t = K\, u_e \delta^*$$

$\hookrightarrow$ velocity - length

$\downarrow (1-U_s)u_e$ ↑ defect $\qquad \downarrow \delta$

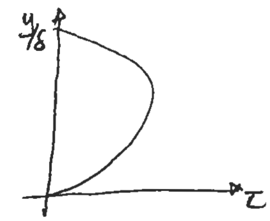
Alternatively

$$\mu_t = K\rho(1-U_s)u_e\delta$$

substituting in assumed profile

$$\tau = \mu_t \frac{\partial u}{\partial y}$$

$$= K\rho u_e^2 \delta^* (1-U_s)\frac{\pi}{2\delta} \sin(\pi y/\delta)$$



$$= K\rho u_e^2 \frac{\pi}{4}(1-U_s)^2 \sin(\pi y/\delta)$$

$$\left(\frac{\delta^*}{\delta} = \frac{1-U_s}{2}\right)$$

Evaluate at max shear $(y/\delta = 1/2)$ (express max shear in terms of profile parameters)

$$\tau_{max} = K\rho u_e^2 \frac{\pi}{4}(1-U_s)^2$$

Define shear stress coefficient

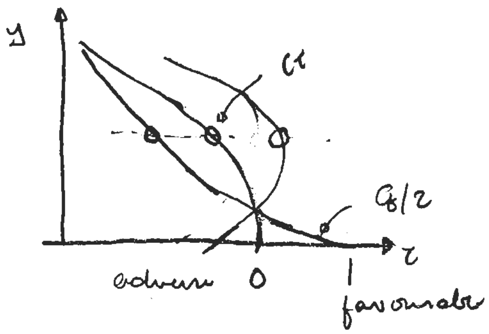
$$C_\tau = \frac{\tau_{max}}{\rho u_e^2} = \frac{\nu_t}{u_e^2}\frac{\partial u}{\partial y}\bigg|_{y_{max}=\delta/2} = \underline{\underline{K\cdot\frac{\pi}{4}(1-U_s)^2}}$$

Relate C_τ to pressure gradient parameter. X-mom eqn is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{\partial \tau}{\partial y}\frac{1}{\rho}$$

At the wall $\vec{u} \to 0$ so

$$\frac{\partial \tau}{\partial y} = \frac{dp}{dx}$$



$$\tau = \tau_w + \frac{\partial \tau}{\partial y} y$$

$$= \tau_w + \frac{dp}{dx} y$$

$$\therefore \quad \frac{\tau}{\rho U_e^2} = \frac{\tau_w}{\rho U_e^2} + \frac{1}{\rho U_e^2}\frac{dp}{dx} y.$$

$$C_\tau = C_f/2 + \underbrace{\frac{1}{\rho U_e^2}\frac{dp}{dx} y_{max}}_{\frac{\delta^*}{\rho U_e^2}\frac{dp}{dx}\frac{y_{max}}{\delta^*}\frac{\tau_w}{\tau_w}}$$

$$\Rightarrow \quad C_f/2 \, \beta \, \frac{y_{max}}{\delta^*}$$

$$\therefore \quad C_\tau = C_f/2\left(1 + \beta \frac{y_{max}}{\delta^*}\right) \; //$$

From G-β relation, recall

$$\frac{H-1}{H} = \frac{3}{4}(1-U_s)$$

$$C_\tau = \frac{\pi}{4} K (1-U_s)^2$$

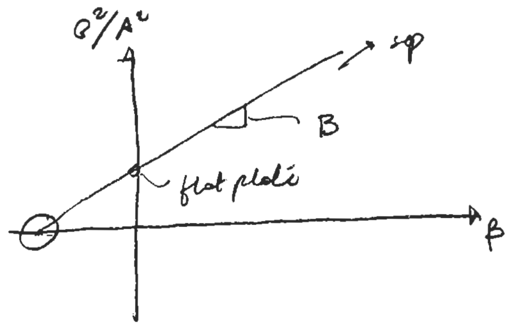
$$G = \sqrt{\frac{2}{C_f}}\left(\frac{H-1}{H}\right) = \sqrt{\frac{2}{C_f}} \cdot \frac{3}{4}(1-U_s)$$

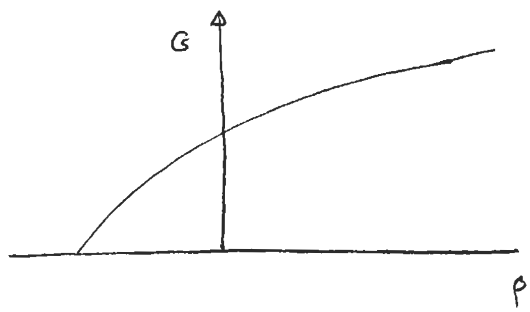
$$\Rightarrow \quad G^2 = \underbrace{\frac{9}{4\pi K}}_{A^2}\underbrace{\frac{C_\tau}{C_f/2}}_{1+B\beta} = A^2(1+B\beta)$$

Therefore G^2 is linear with β.

$$A \approx \frac{3}{2}\frac{1}{\sqrt{\pi K}} \approx 6.5 - 6.93$$

$$B \approx y_{max}/\delta^* \approx 0.7 - 0.75$$

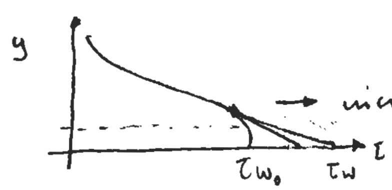




Quantitative relationship between G, $\frac{dp}{dx}$, H $\rightarrow$ required for turbulent closure.

— X —

Suction effect.



$\tau \approx \tau_w + \rho v_w U$, outer layer unaffected.

$\tau_w \approx \tau_{w_0} - \rho v_w U \qquad v_w < 0 \Rightarrow$ suction