4-4) Integral Methods

A) Integral Momentum Eqn

B) Thwaites Method

Reading:

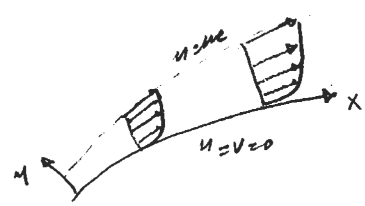
White - 264. - 274.
Sch - 191 - 202 (682-698 opt)
C&B - 104 - 116
(Mom Trans in BL)

Handouts: Thwaites method (2)
Basis for int methods (1)
Eqn of BL thickness (1)
4

A) Integral Method

Goal. Solve non-similar boundary layer: $u_e(x)$ arbitrary

Typical problem:



Can be solved by:

1. F. Diff. method
   - relatively numerically intensive
   - solve PDE in x, y.

2. Integral method
   - more economical/efficient
   - ODE in x only
   - provides insight into physical behavior

Relative Cost:

	Known u_e	Unknown U_e	
1	10	> 1000	
2	1	50	(inverse problem)

Integral Momentum Equation

$$(u - u_e)\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right]$$

$$+ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - u_e\frac{du_e}{dx} - \frac{1}{\rho}\frac{\partial \tau}{\partial y} = 0$$

Integrate from $0 \rightarrow y_e(x)$ $\rightarrow$ wash out details in y, convert PDE in x,y to ODE in x

$$\int_0^{y_e}\left\{(u-u_e)\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - u_e\frac{du_e}{dx} - \frac{1}{\rho}\frac{\partial \tau}{\partial y}\right\} dy = 0$$

$$\rightarrow \int_0^{y_e}\left\{\frac{\partial}{\partial x}\left[(u_e-u)u\right] + \frac{\partial}{\partial y}\left[(u_e-u)v\right] + (u_e-u)\frac{du_e}{dx} + \frac{1}{\rho}\frac{\partial \tau}{\partial y}\right\} dy = 0$$

$$\frac{d}{dx}\int_0^{y_e}\left[(u_e-u)u\right]dy + 0 + \frac{du_e}{dx}\int(u_e-u)\,dy - \frac{\tau_w}{\rho} = 0$$

$$\frac{d}{dx}\left(\rho u_e^2\theta\right) + \rho u_e\delta^*\frac{du_e}{dx} = \tau_w \quad \leftarrow \text{dimensional form}$$

$$\frac{d\theta}{dx} + (H+2)\frac{\theta}{u_e}\frac{du_e}{dx} = C_f/2 \quad \leftarrow \text{non-dim form}$$

where $\delta^* = \int(1 - u/u_e)\,dy$, $\theta = \int(1-u/u_e)(u/u_e)\,dy$, $H = \delta^*/\theta$

$C_f = \tau_w / \tfrac{1}{2}\rho U_e^2$

Other forms

Logarithmic:

$$\frac{d}{dx}(\ln\theta) = \frac{1}{\theta}\,C_f/2 - (H+2)\frac{d}{dx}(\ln u_e) \qquad \text{— div. by } \theta$$

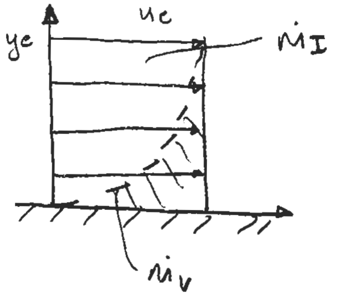
Similarity form:

$$\beta_\theta = \left(\frac{x}{\theta}\right) C_f/2 - (H+2)\,\beta_u$$

with $$\beta_{()} = \frac{x}{()}\frac{d()}{dx} = \frac{d(\ln())}{d(\ln x)}$$

Physical Interpretation

a)

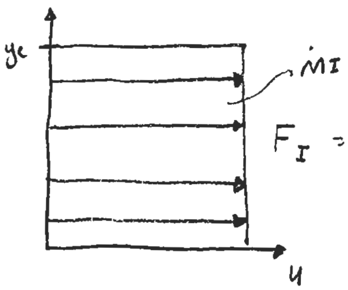


$$\dot{m} = \int_0^{y_e} d\dot{m} = \int_0^{y_e} \rho u \, dy$$

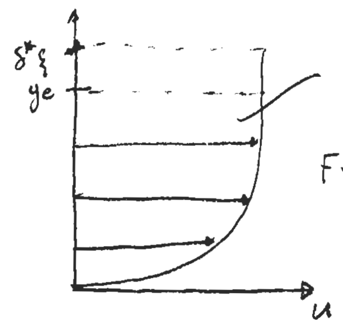
$$\dot{m}_I = \rho u_e y_e$$

$$\dot{m}_V = \dot{m}_I - \rho u_e \delta^*$$

b)



$$F_I = \rho u_e^2 y_e$$



$$\dot{m}_I = \dot{m}_V = \int_0^{y_e+\delta^*} \rho u \, dy$$

$$F_V = \int_0^{y_e+\delta^*} \rho u^2 \, dy$$

$$= F_I - \rho u_e^2 \theta$$

profile drag.

solve, we must integrate VKI mom equation

$$\theta(x) = \int_{x_0}^{x} \left(C_f/2 - (H+2)\frac{\theta}{u_e}\frac{du_e}{dx} \right) dx + \theta_0$$

- Need $C_f(x)$ and $H(x)$ to integrate, assuming u_e is given,
- All integral methods make approximation for $C_f(x)$ and $H(x)$

Ex:

$$C_f = C_f(U_e, \theta)$$
$$H = C_f(u_e, \theta)$$

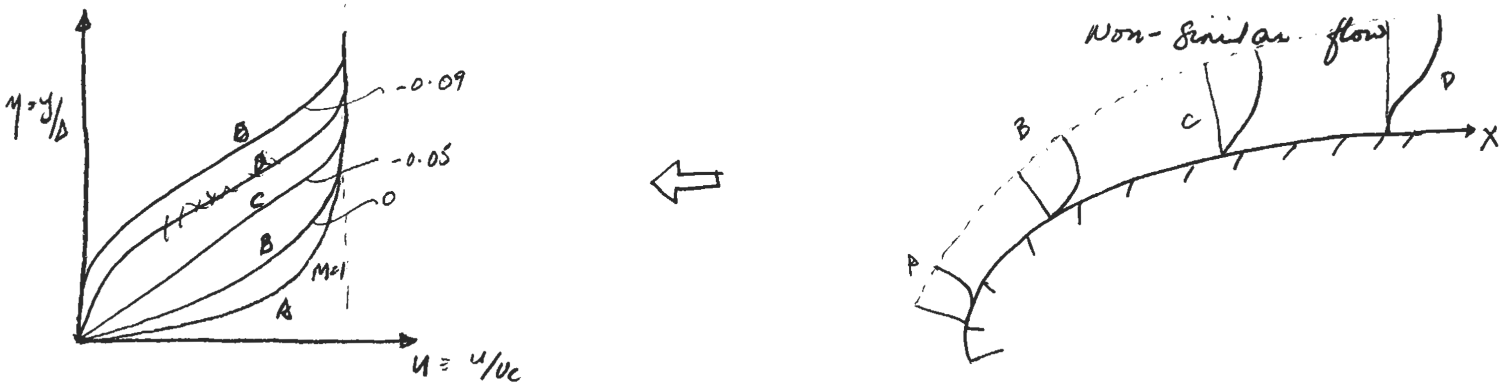
Basis for Integral BL Methods

Underlying Assumption: $u(y)$ at any x can be represented by a profile family with suitable rescaling of u and y.

- analogous to modal or spectral representation
$u = \sum A_k \sin(\pi k (y/y_e))$

Example: Falkner-Skan profiles

$$U(\eta; m) \text{ or } U(\eta; H) \quad \text{— one parameter family}$$



To fit profile at x, we need in general

anything that uniquely identifies the profile - $\delta^*, \theta, \ldots$

- Suitable profile parameter $\rightarrow$ non-dim pressure gradient (force, dynamic quantity)
- Normal length scale $\rightarrow$ Δ
- Velocity scale (typically u_e $\rightarrow$ $u = U u_e$.

These should be "local" scaling parameters

B) Thwaites' Method:

- Applicable to laminar, incompressible, BL development
- One equation method — uses only VKI momentum eqn.

Pick normal length scale $\Delta = \theta$

velocity scale u_e

profile parameter $\dfrac{-dp/dx}{\partial\tau/\partial y} \sim \dfrac{-dp/dx}{\mu U_e/\theta^2} = \dfrac{\theta^2}{\nu}\dfrac{du_e}{dx} \equiv \lambda$ (Thwaites' parameter)

$\uparrow \; \dfrac{\partial \tau}{\partial y} \sim O\left(\mu \dfrac{u_e}{\theta^2}\right)$

locity profiles are characterized by

a) $\left.\dfrac{d^2U}{d\eta^2}\right|_0$ — curvature at wall (related to $\dfrac{du_e}{dx}$) $= -\dfrac{\theta^2}{\nu}\dfrac{du_e}{dx} \equiv -\lambda$

b) $\left.\frac{du}{d\eta}\right|_0$ profile slope at wall (related to C_f) : $\frac{u_e\theta}{\nu} C_f/2 = \ell$

c) $H = \delta^*/\theta$ shape parameter

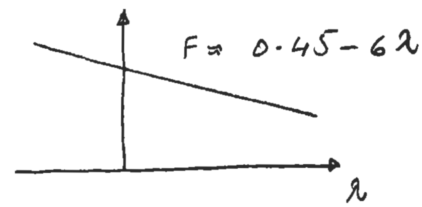
Thwaite's assumption: ℓ and H depend only on $\lambda \rightarrow \ell = \ell(\lambda), H = H(\lambda)$

Substituting in VKI ...

$$\frac{d\theta}{dx} = \frac{\nu}{u_e\theta}\ell - (H+2)\frac{\nu}{u_e\theta}\lambda$$

or $$\frac{u_e}{\nu}\frac{d}{dx}(\theta^2) = 2\left[\ell - (H+2)\lambda\right] \equiv F(\lambda)$$

Curve fit $F(\lambda) \rightarrow$ is nearly linear



$$\therefore \quad \frac{1}{\nu}\frac{d}{dx}\left(u_e^6\theta^2\right) = 0.45\, u_e^5 \text{ — int. fact.}$$

$$\rightarrow \quad u_e^6\theta^2 - u_e^6(x_0)\,\theta(x_0)^2 = 0.45\,\nu\int_{x_0}^{x} u_e(t)^5\,dt$$

Given $u_e(x)$ we can integrate to get $\theta(x)$
using forward Euler (for example)

$$\theta(x)^2 = \frac{1}{u_e(x)^6}\left[u_e(x_0)^6\,\theta(x_0)^2 + 0.45\,\nu\int_{x_0}^{x} u_e(t)^5\,dt\right]$$

We can $\lambda(x) = \frac{\theta^2(x)}{\nu}\frac{du_e(x)}{dx}$, $\ell(x) = \ell(\lambda(x))$, $H(x) = H(\lambda(x))$

curve fit

$\delta^*(x) = H(x)\,\theta(x)$

$C_f(x) = \frac{2\nu}{u_e(x)\theta(x)}\ell(x)$