Appendix B
Closure for Three-Dimensional Boundary Layer Equations

B.1 1-2 Coordinate Definitions

$$1 \Longrightarrow \text{Streamwise Direction}$$

$$2 \Longrightarrow \text{Crossflow Direction}$$

$$\theta_{11} = \int \left(1 - \frac{u_1}{q_e}\right) \frac{\rho}{\rho_e} \frac{u_1}{q_e} d\eta \qquad \theta_{12} = \int \left(1 - \frac{u_1}{q_e}\right) \frac{\rho}{\rho_e} \frac{u_2}{q_e} d\eta$$

$$\theta_{21} = \int \left(-\frac{u_2}{q_e}\right) \frac{\rho}{\rho_e} \frac{u_1}{q_e} d\eta \qquad \theta_{22} = \int \left(-\frac{u_2}{q_e}\right) \frac{\rho}{\rho_e} \frac{u_2}{q_e} d\eta$$

$$\delta_1^* = \int 1 - \frac{\rho}{\rho_e} \frac{u_1}{q_e} d\eta \qquad \delta_2^* = \int - \frac{\rho}{\rho_e} \frac{u_2}{q_e} d\eta$$

$$\theta_\rho = \int 1 - \frac{\rho}{\rho_e} d\eta$$

$$\delta_1^{**} = \int \left(1 - \frac{\rho}{\rho_e}\right) \frac{u_1}{q_e} d\eta \qquad \delta_2^{**} = \int \left(1 - \frac{\rho}{\rho_e}\right) \frac{u_2}{q_e} d\eta$$

$$E_{11} = \int \left[1 - \left(\frac{u_1}{q_e}\right)^2\right] \frac{\rho}{\rho_e} \frac{u_1}{q_e} d\eta \qquad E_{12} = \int \left[1 - \left(\frac{u_1}{q_e}\right)^2\right] \frac{\rho}{\rho_e} \frac{u_2}{q_e} d\eta$$

$$E_{21} = \int - \left(\frac{u_2}{q_e}\right)^2 \frac{\rho}{\rho_e} \frac{u_1}{q_e} d\eta \qquad E_{22} = \int - \left(\frac{u_2}{q_e}\right)^2 \frac{\rho}{\rho_e} \frac{u_2}{q_e} d\eta$$

$$\theta_1^* = E_{11} + E_{21} \qquad \theta_2^* = E_{12} + E_{22}$$

B.2 x-z Coordinate Definitions

Using a rotation matrix, the x-z thicknesses may be determined in terms of the 1-2 thicknesses and the angle between the two coordinate systems ($\cos\alpha = \frac{u_e}{q_e}, \sin\alpha = \frac{w_e}{q_e}$) [38, 49].

$$\rho_e q_e^2 \theta_{xx} = \rho_e u_e^2 \theta_{11} + \rho_e w_e^2 \theta_{22} - \rho_e u_e w_e (\theta_{12} + \theta_{21})$$

$$\rho_e q_e^2 \theta_{xz} = \rho_e u_e^2 \theta_{12} - \rho_e w_e^2 \theta_{21} + \rho_e u_e w_e (\theta_{11} - \theta_{22})$$

$$\rho_e q_e^2 \theta_{zx} = \rho_e u_e^2 \theta_{21} - \rho_e w_e^2 \theta_{12} + \rho_e u_e w_e (\theta_{11} - \theta_{22})$$

$$\rho_e q_e^2 \theta_{zz} = \rho_e u_e^2 \theta_{22} + \rho_e w_e^2 \theta_{11} + \rho_e u_e w_e (\theta_{12} + \theta_{21})$$

$$\rho_e q_e \delta_x^* = \rho_e u_e \delta_1^* - \rho_e w_e \delta_2^*$$

$$\rho_e q_e \delta_z^* = \rho_e u_e \delta_2^* + \rho_e w_e \delta_1^*$$

$$\rho_e q_e^2 \delta_x^{**} = \rho_e u_e \delta_1^{**} - \rho_e w_e \delta_2^{**}$$

$$\rho_e q_e^2 \delta_z^{**} = \rho_e u_e \delta_2^{**} + \rho_e w_e \delta_1^{**}$$

$$\rho_e q_e^3 \theta_x^* = q_e^2 (\rho_e u_e \theta_1^* - \rho_e w_e \theta_2^*)$$

$$\rho_e q_e^3 \theta_z^* = q_e^2 (\rho_e u_e \theta_2^* + \rho_e w_e \theta_1^*)$$

$$\tau_{xw} = \frac{u_e}{q_e}\tau_1 - \frac{w_e}{q_e}\tau_2$$

$$\tau_{zw} = \frac{u_e}{q_e}\tau_2 + \frac{w_e}{q_e}\tau_1$$

$$D = \int \tau_1 du_1 + \int \tau_2 du_2$$

B.3 Crossflow Model

The crossflow model is Johnston's triangular profile [28]

$$\frac{u_2}{q_e} = A_c\left(1 - \frac{u_1}{q_e}\right) \tag{B.1}$$

where A_c is the crossflow parameter. Streamwise-crossflow thicknesses may now be defined.

B.4 Derived Thicknesses

$$
\begin{aligned}
\delta_2^* &= \int -\frac{\rho}{\rho_e}\frac{u_2}{q_e}\,d\eta \\
&= \int -A_c\left(1 - \frac{u_1}{q_e}\right)\frac{\rho}{\rho_e}\,d\eta \\
&= \int -A_c\left(1 - \frac{\rho}{\rho_e}\frac{u_1}{q_e}\right)d\eta + \int A_c\left(1 - \frac{\rho}{\rho_e}\right)d\eta \\
&= A_c\left(\theta_\rho - \delta_1^*\right)
\end{aligned}
\tag{B.2}
$$

$$
A_c = \frac{\delta_2^*}{\theta_{11}}\,\frac{1}{H_{\theta_\rho} - H}
\tag{B.3}
$$

$$
\begin{aligned}
\theta_{21} &= \int -\frac{u_2}{q_e}\frac{\rho}{\rho_e}\frac{u_1}{q_e}\,d\eta \\
&= \int -A_c\left(1 - \frac{u_1}{q_e}\right)\frac{\rho}{\rho_e}\frac{u_1}{q_e}\,d\eta \\
&= -A_c\theta_{11}
\end{aligned}
\tag{B.4}
$$

$$
\begin{aligned}
\theta_{12} &= \int \left(1 - \frac{u_1}{q_e}\right)\frac{\rho}{\rho_e}\frac{u_2}{q_e}\,d\eta \\
&= \int -\frac{u_2}{q_e}\frac{\rho}{\rho_e}\frac{u_1}{q_e}\,d\eta + \int \frac{\rho}{\rho_e}\frac{u_2}{q_e}\,d\eta \\
&= \theta_{21} - \delta_2^*
\end{aligned}
\tag{B.5}
$$

$$
\begin{aligned}
\theta_{22} &= \int -\frac{u_2}{q_e}\frac{\rho}{\rho_e}\frac{u_2}{q_e}\,d\eta \\
&= \int -A_c\left(1 - \frac{u_1}{q_e}\right)\frac{\rho}{\rho_e}\frac{u_2}{q_e}\,d\eta \\
&= -A_c\theta_{12}
\end{aligned}
\tag{B.6}
$$

$$\begin{aligned}
\delta_2^{**} &= \int \left(1 - \frac{\rho}{\rho_e}\right) \frac{u_2}{q_e}\, d\eta \\
&= \int \left(1 - \frac{\rho}{\rho_e}\right) A_c \left(1 - \frac{u_1}{q_e}\right) d\eta \qquad \text{(B.7)} \\
&= A_c \left[\theta_\rho - \delta_1^{**}\right]
\end{aligned}$$

$$E_{12} = \theta_{12} + A_c\,(2\theta_{11} - E_{11}) \qquad \text{(B.8)}$$

$$E_{21} = -\theta_{22} - A_c E_{12} \qquad \text{(B.9)}$$

$$E_{22} = -A_c\,(E_{21} - \theta_{22}) \qquad \text{(B.10)}$$

B.5 Empirical Closure Relations

The following closure relations are taken from subroutines of Drela. The references for these relations are labeled along with the formula.

Shape Parameters

$$\begin{aligned}
H &\equiv \frac{\delta_1^*}{\theta_{11}} \\
H_{\theta_\rho} &\equiv \frac{\theta_\rho}{\theta_{11}} \\
H_{\delta_{11}^{**}} &\equiv \frac{\delta_1^{**}}{\theta_{11}} \qquad \text{(B.11)} \\
H^* &\equiv \frac{E_{11}}{\theta_{11}}
\end{aligned}$$