
Singular Perturbation Method

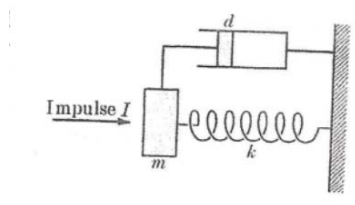
What formal analytical methods do we apply to solve "boundary layer" problems?

What formal analytical methods do we apply to solve differential equations where in the limit of a small parameter vanishing, one or more highest-order derivative terms drop out?

What formal analytical methods do we apply to solve problems where competing physical mechanisms vary over time?

To "answer" the above questions, we introduce the singular perturbation method. In applying the singular perturbation method (SPM), one should 1. focus on competing physical mechanisms, 2. identify a small parameter, and/or 3. observe that the governing differential equation degenerates in the limit of a small parameter vanishing.

Consider the mass-string-damper system shown in the figure below.



m = body mass, constant
 d = viscous damping constant
 k = spring constant
 I = applied impulse, at $t = 0$
 t = time
 x = displacement

Governing equation and boundary conditions:

$$m \frac{d^2 x}{dt^2} + d \frac{dx}{dt} + kx = 0$$

$$x(0) = 0$$

$$u(0) = \left(\frac{dx}{dt} \right)_{t=0} = \frac{I}{m}$$

The competing physical mechanisms are:

1. Acceleration/convective forces
2. Damping forces
3. Spring constant based forces

Exact solution:

$$x = \left(\frac{I}{d}\right) \left(e^{-\lambda_2 t} - e^{-\lambda_1 t}\right) \sqrt{1 - 4 \frac{mk}{d^2}}$$

$$\lambda_{1,2} = -\frac{d}{2m} \left(1 \pm \sqrt{1 - 4 \frac{mk}{d^2}}\right)$$

For large I and small m , the competing forces are acceleration forces and damping forces in the initial stages (small times).

For large times, after the mass has reached its maximum deviation, the competing forces are damping and spring constant.

Solutions valid for small times are called **inner solutions**.

Solutions valid for large times are called **outer solutions**.

Note for the mass-spring-damper problem:

$$\text{Inner solution: } x^i = \frac{I}{d} \left[1 - e^{-(d/m)t}\right]$$

$$\text{Outer solution: } x^o = \frac{I}{d} e^{-(k/d)t}$$

We now apply SPM to solve the above problem. First, introduce dimensionless variables:

$$t^* = \frac{k}{d} t \quad x^* = \frac{d}{I} x$$

Substitute and obtain:

$$\epsilon \frac{d^2 x^*}{dt^{*2}} + \frac{dx^*}{dt^*} + x^* = 0$$

$$\left(\frac{dx^*}{dt^*}\right)_{t^*=0} = \frac{1}{\epsilon}$$

$$x^*(0) = 0$$

$$\epsilon = \frac{mk}{d^2} \ll 1$$

Consider an outer expansion of the form:

$$x^* = x^0 = \sum x_n^0 \epsilon^n$$

Substitute and equate terms of like powers of ϵ :

$$\frac{dx_0^0}{dt^*} + x_0^0 = 0$$

$$\frac{dx_n^0}{dt^*} + x_n^0 = -\frac{d^2 x_{n-1}^0}{dt^{*2}}$$

The solution for x_0^0 is:

$$x_0^0 = A_0^0 e^{-t^*}$$

$A_0^0 = \text{constant}$, to be determined

Consider an inner solution (small times). In order “to see” in inner scale, we will “magnify” the region of interest.

$$x^* = x^i$$

$$\bar{t} = \frac{t^*}{\epsilon}, \text{ magnification of } t^*$$

Substitute and obtain

$$\frac{d^2 x^i}{d\bar{t}^2} + \frac{dx^i}{d\bar{t}} + \epsilon x^i = 0$$

$$\left(\frac{dx^i}{d\bar{t}}\right)_{\bar{t}=0} = 1$$

$$x^i(0) = 0$$

Consider an inner expansion of the form:

$$x^i = \sum x_n^i(\bar{t}) e^n$$

Substitute and obtain:

$$\frac{d^2 x_0^i}{d\bar{t}^2} + \frac{dx_0^i}{d\bar{t}} = 0$$

$$\frac{d^2 x_n^i}{d\bar{t}^2} + \frac{dx_n^i}{d\bar{t}} = -x_{n-1}^i, n > 0$$

[Question: why not $\bar{t} = t^*/\epsilon^2$ or $\bar{t} = t^*/\epsilon^3$?]

We obtain x_0^i as

$$x_0^i = 1 - e^{-\bar{t}}$$

Now let's determine A_0^0 . Assume an overlap region exists where the inner and outer solutions are valid. The overlap region exists in t^* , we seek that region in which

$$t^* = \delta(\epsilon)$$

$$\lim_{\epsilon \rightarrow 0} \delta(\epsilon) = 0$$

and

$$\lim_{\epsilon \rightarrow 0} \left(\frac{\delta(\epsilon)}{\epsilon}\right) = \infty$$

We select

$$\delta(\epsilon) = \sqrt{\epsilon}$$

hence

$$\lim_{\epsilon \rightarrow 0} \left[x_0^i \left(\frac{\delta(\epsilon)}{\epsilon} \right) \right] = \lim_{\epsilon \rightarrow 0} \left[x_0^i(\delta(\epsilon)) \right]$$

or

$$x_0^i(\infty) = x_0^i(0)$$

The above two equations illustrate the limit matching principle: the outer limit of the inner expansion = the inner limit of the outer expansion.

Apply the limit matching principle [LMP]:

$$x_0^i(\infty) = x_0^{i0} = 1$$

$$x_0^0(0) = x_0^{0i} = A_0^0$$

$$x_0^{i0} = x_0^{0i}$$

$$A_0^0 = 1$$

The composite solution that is uniformly valid to order ϵ over the whole region is defined as x_0^c .

$$x_0^c \sim x_0^0 + x_0^i - x_0^{0i} \quad (0.1)$$

$$\sim e^{-t^*} + (1 - e^{-\bar{t}}) - 1 \quad (0.2)$$

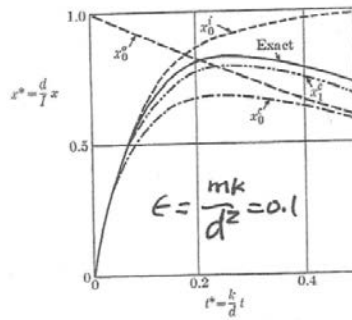
$$\sim e^{-t^*} - e^{-\bar{t}} = e^{-t^*} - e^{-t^*/\epsilon} \quad (0.3)$$

To continue the analysis to compute x_1^i and x_1^0 , we will need the asymptotic matching principle [AMP]:

The m-term outer expansion of [the n-term inner expansion] = the n-term inner expansion of [the m-term outer expansion]

One may show that a composite solution that is uniformly valid to order ϵ^2 over the whole region is:

$$x^c \sim x^i + x^0 - x^{i0} = (1 + 2\epsilon)(e^{-t^*} - e^{-t^*/\epsilon}) - t^*(e^{-t^*/\epsilon} + \epsilon e^{-t^*})$$



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