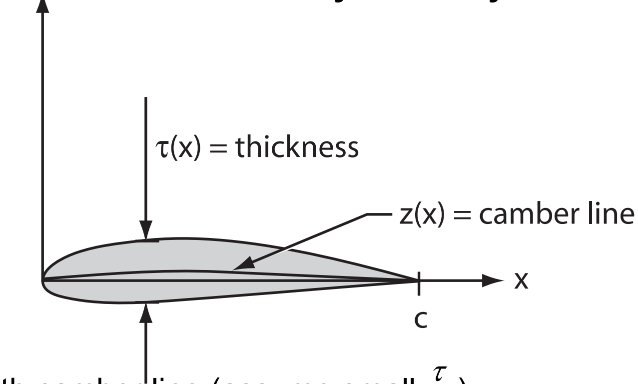
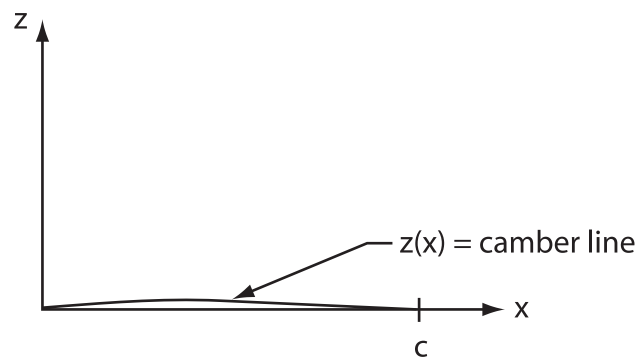
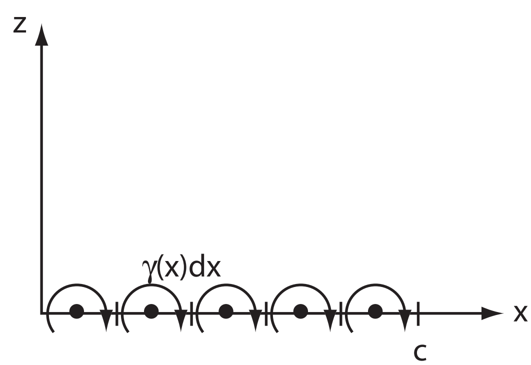
Thin Airfoil Theory Summary



- Replace airfoil with camber line (assume small $\frac{\tau}{c}$)



- Distribute vortices of strength $\gamma(x)$ along chord line for $0 \le x \le c$.



- Determine $\gamma(x)$ by satisfying flow tangency on camber line.

$$V_\infty\left(\alpha - \frac{dZ}{dx}\right) - \int_0^c \frac{\gamma(\xi)d\xi}{2\pi(x-\xi)} = 0$$

- The pressure coefficient can be simplified using Bernoulli & assuming small perturbation:

$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2}$$

$$p + \frac{1}{2}\rho\left\{(V_\infty + \tilde{u})^2 + \tilde{V}^2\right\} = p_\infty + \frac{1}{2}\rho V_\infty^2$$

$$\Rightarrow \frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2} = 1 - \frac{(V_\infty + \tilde{u})^2 + \tilde{V}^2}{V_\infty^2}$$

$$= 1 - \frac{V_\infty^2 + 2V_\infty \tilde{u} + \tilde{u}^2 + \tilde{V}^2}{V_\infty^2}$$

$$= -2\frac{\tilde{u}}{V_\infty} - \underbrace{\frac{\tilde{u}^2 + \tilde{V}^2}{V_\infty^2}}_{\text{higher order}}$$

$$\Rightarrow \boxed{C_p = -2\frac{\tilde{u}}{V_\infty}}$$

- It can also be shown that

$$\gamma(x) = \tilde{u}_{upper}(x) - \tilde{u}_{lower}(x)$$

$$\Rightarrow \Delta C_p = C_{p_{lower}} - C_{p_{upper}} = \frac{2}{V_\infty}(\tilde{u}_{upper} - \tilde{u}_{lower})$$

$$\Rightarrow \boxed{C_p(x) = 2\frac{\gamma(x)}{V_\infty}}$$

<u>Symmetric Airfoil Solution</u>

For a symmetric airfoil (i.e. $\frac{dz}{dx} = 0$), the vortex strength is:

$$\gamma(\theta) = 2\alpha V_\infty \frac{1 + \cos\theta}{\sin\theta}$$

But, recall:

$$x = \frac{c}{2}(1 - \cos\theta)$$

$$\Rightarrow \cos\theta = 1 - 2\frac{x}{c}$$

$$\sin\theta = \sqrt{1-\cos^2\theta}$$

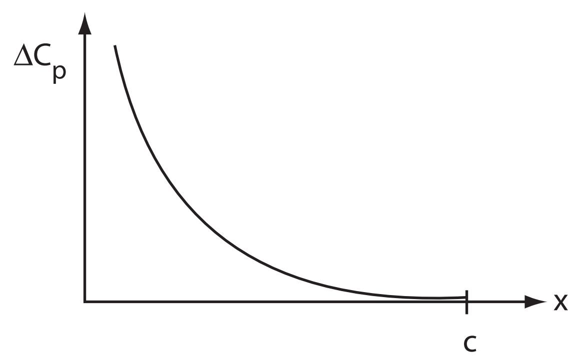
$$= \sqrt{1-\left(1-2\frac{x}{c}\right)^2}$$

$$\sin\theta = 2\sqrt{\frac{x}{c}(1-\frac{x}{c})}$$

$$\Rightarrow \gamma(x) = 2\alpha V_\infty \frac{1-\frac{x}{c}}{\sqrt{\frac{x}{c}(1-\frac{x}{c})}}$$

$$\boxed{\gamma(x) = 2\alpha V_\infty \sqrt{\frac{1-\frac{x}{c}}{\frac{x}{c}}}}$$

Thus, $\Delta C_p = 4\alpha\sqrt{\frac{1-\frac{x}{c}}{\frac{x}{c}}}$.



Some things to notice:

- At trailing edge $\Delta C_p = 0$.

  $\Rightarrow$ Kutta condition is enforced which requires $p_{upper} = p_{lower}$
- At leading edge, $\Delta C_p \to \infty$! "Suction peak" required to turn flow around leading edge which is infinitely thin.

  The instance of a suction peak exists on true airfoils (i.e. not infinitely thin) though ΔC_p is finite (but large).

Suction peaks should be avoided as they can result in

1. leading edge separation
2. low (very low) pressure at leading edge which must rise towards trailing edge $\Rightarrow$ adverse pressure gradients $\Rightarrow$ boundary layer separation.

Cambered Airfoil Solutions

For a cambered airfoil, we can use a "Fourier series"–like approach for the vortex strength distribution:

$$\Rightarrow \gamma(\theta) = 2V_\infty \left[\underbrace{A_o \frac{1+\cos\theta}{\sin\theta}}_{\text{flat plate contributions}} + \underbrace{\sum_{n=1}^{\infty} A_n \sin n\theta}_{\text{cambered contributions}} \right]$$

Plugging this into the flow tangency condition for the camber line gives (after some work):

$$A_0 = \alpha - \frac{1}{\pi}\int_0^\pi \frac{dz}{dx} d\theta_0$$

$$A_n = \frac{2}{\pi}\int_0^\pi \frac{dz}{dx} \cos n\theta_0 d\theta_0$$

After finding the A_n's, the following relationships can be used to find $C_\ell, C_{m_{ac}}$, etc.

$$C_\ell = 2\pi(\alpha - \alpha_{LO})$$

$$C_{m_{c/4}} = C_{m_{ac}} = \frac{\pi}{4}(A_2 - A_1)$$

$$\alpha_{LO} = -\frac{1}{\pi}\int_0^\pi \frac{dz}{dx}(\cos\theta_0 - 1) d\theta_0 = \alpha - A_0 - \frac{1}{2}A_1$$

Note: in thin airfoil theory, the aerodynamic center is always at the quarter-chord ($c/4$), regardless of the airfoil shape or angle of attack.