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16.07 Dynamics
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Lecture L13 - Conservative Internal Forces and Potential Energy

The forces internal to a system are of two types. Conservative forces, such as gravity; and dissipative forces such as friction. Internal forces arise from the natural dynamics of the system in contract to external forces which are imposed from an external source.

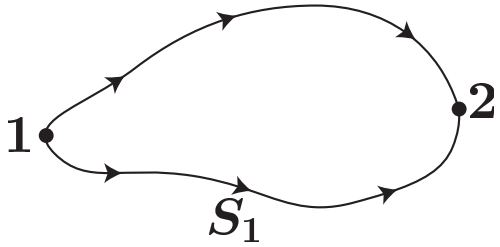
We have seen that the work done by a force $\boldsymbol{F}$ on a particle is given by $dW = \boldsymbol{F} \cdot d\boldsymbol{r}$.

If the work done by an internal forces $\boldsymbol{F}$, when the particle moves from any position $\boldsymbol{r}_1$ to any position $\boldsymbol{r}_2$, can be expressed as the difference in a scalar function of r between the two ends of the trajectory,

$$W_{12} = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot d\boldsymbol{r} = -(V(\boldsymbol{r}_2) - V(\boldsymbol{r}_1)) = V_1 - V_2 \ , \tag{1}$$

then we say that the force is *conservative*. In the above expression, the scalar function $V(\boldsymbol{r})$ is called the *potential*. It is clear that the potential satisfies $dV = -\boldsymbol{F} \cdot d\boldsymbol{r}$ (the minus sign is included for convenience).

There are two main consequences that follow from the existence of a potential: i) the work done by a conservative force between points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ is *independent of the path*. This follows from (1) since W_{12} only depends on the initial and final potentials V_1 and V_2 (and not on how we go from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$), and ii) the work done by potential forces is *recoverable*. Consider the work done in going from point $\boldsymbol{r}_1$ to point $\boldsymbol{r}_2$, W_{12}. If we go, now, from point $\boldsymbol{r}_2$ to $\boldsymbol{r}_1$, we have that $W_{21} = -W_{12}$ since the total work $W_{12} + W_{21} = (V_1 - V_2) + (V_2 - V_1) = 0$.



In one dimension any force which is only a function of position is conservative. That is, if we have a force, $F(x)$, which is only a function of position, then $F(x)\,dx$ is always a perfect differential. This means that we can define a potential function as

$$V(x) = -\int_{x_0}^{x} F(x)\,dx \ ,$$

where x_0 is arbitrary.

In two and three dimensions, we would, in principle, expect that any force which depends only on position, $\boldsymbol{F}(\boldsymbol{r})$, to be conservative. However, it turns out that, in general, this is not sufficient. In multiple dimensions,

the condition for a force field to be conservative is that it can be expressed as the gradient of a potential function. That is,

$$\boldsymbol{F}_C = -\boldsymbol{\nabla} V \ .$$

This result follows from the gradient theorem, which is often called the fundamental theorem of calculus, which states that the integral

$$-\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{\nabla} V \cdot d\boldsymbol{r} = -(V_2 - V_1)$$

is independent of the path between r_1 and r_2. Therefore the work done by conservative forces depends only upon the endpoints r_2 and r_1 rather than the details of the path taken between them.

$$\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F}_C \cdot d\boldsymbol{r} = -\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{\nabla} V \cdot d\boldsymbol{r} = -(V_2 - V_1)$$

In the general case, we will deal with internal forces that are a combination of conservative and non-conservative forces.

$$\boldsymbol{F} = \boldsymbol{F}_C + \boldsymbol{F}_{NC} = -\boldsymbol{\nabla} V + \boldsymbol{F}_{NC} \ .$$

Note ***The gradient operator, $\boldsymbol{\nabla}$***

The gradient operator, $\boldsymbol{\nabla}$ (called "del"), in cartesian coordinates is defined as

$$\boldsymbol{\nabla}(\ ) \equiv \frac{\partial(\ )}{\partial x}\boldsymbol{i} + \frac{\partial(\ )}{\partial y}\boldsymbol{j} + \frac{\partial(\ )}{\partial z}\boldsymbol{k} \ .$$

When operating on a scalar function $V(x, y, z)$, the result $\boldsymbol{\nabla} V$ is a vector, called the gradient of V. The components of $\boldsymbol{\nabla} V$ are the derivatives of V along each of the coordinate directions,

$$\boldsymbol{\nabla} V \equiv \frac{\partial V}{\partial x}\boldsymbol{i} + \frac{\partial V}{\partial y}\boldsymbol{j} + \frac{\partial V}{\partial z}\boldsymbol{k} \ .$$

If we consider a particle moving due to conservative forces with potential energy $V(x, y, z)$, as the particle moves from point $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ to point $\boldsymbol{r} + d\boldsymbol{r} = (x + dx)\boldsymbol{i} + (y + dy)\boldsymbol{j} + (z + dz)\boldsymbol{k}$, the potential energy changes by $dV = V(x + dx, y + dy, z + dz) - V(x, y, z)$. For small increments dx, dy and dz, and dV, can be expressed, using Taylor series expansions, as

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = \boldsymbol{\nabla} V \cdot d\boldsymbol{r} \ ,$$

where $d\boldsymbol{r} = dx\boldsymbol{i} + dy\boldsymbol{j} + dz\boldsymbol{k}$.

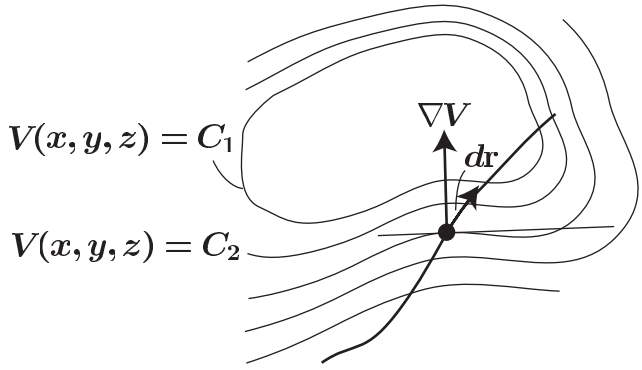
This equation expresses the fundamental property of the gradient. The gradient allows us to find the change in a function induced by a change in its variables.

If we write $V(x, y, z) = C$, for some constant C, this is the implicit equation of a surface, which is called a constant energy surface. This surface is made up by all the points in the x, y, z space for which the function $V(x, y, z)$ is equal to C. It is clear that if a particle moves on a constant energy surface, $dV = 0$, since V is

constant on that surface. Therefore, when a particle moves on a constant energy surface, $d\boldsymbol{r}$ will be tangent to that surface, and since

$$0 = dV = \boldsymbol{\nabla} V \cdot d\boldsymbol{r} \ ,$$

we have that $\boldsymbol{\nabla} V$ is perpendicular to any tangent to the surface. This situation is illustrated in the picture below for the two dimensional case. Here, the constant energy surfaces are contour curves, and we can see that the gradient vector is always normal to the contour curves.



Note **Gradient operator in cylindrical coordinates**

The gradient operator can be expressed in cylindrical coordinates by writing $x = r\cos\theta$, $y = r\sin\theta$, and $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}(y/x)$. Thus, applying the chain rule for differentiation, we have

$$\begin{aligned} \frac{\partial(\ )}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial(\ )}{\partial r} + \frac{\partial r}{\partial x}\frac{\partial(\ )}{\partial \theta} = \cos\theta\frac{\partial(\ )}{\partial r} - \frac{\sin\theta}{r}\frac{\partial(\ )}{\partial r} \\ \frac{\partial(\ )}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial(\ )}{\partial r} + \frac{\partial r}{\partial y}\frac{\partial(\ )}{\partial \theta} = \sin\theta\frac{\partial(\ )}{\partial r} + \frac{\cos\theta}{r}\frac{\partial(\ )}{\partial r} \ . \end{aligned}$$

If we note that $\boldsymbol{i} = \cos\theta\boldsymbol{e}_r - \sin\theta\boldsymbol{e}_\theta$ and $\boldsymbol{j} = \sin\theta\boldsymbol{e}_r + \cos\theta\boldsymbol{e}_\theta$, we have that

$$\boldsymbol{\nabla}(\ ) \equiv \frac{\partial(\ )}{\partial r}\,\boldsymbol{e}_r + \frac{1}{r}\frac{\partial(\ )}{\partial \theta}\,\boldsymbol{e}_\theta + \frac{\partial(\ )}{\partial z} \ .$$

An expression for spherical coordinates can be derived in a similar manner.

Conservation of Energy

When all the forces doing work are conservative, the work is given by (1), and the principle of work and energy derived in the last lecture,

$$T_1 + W_{12} = T_2 \ ,$$

reduces to,

$$T_1 + V_1 = T_2 + V_2$$

or more generally, since the points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are arbitrary,

$$E = T + V = \text{constant} \ . \tag{2}$$

Whenever applicable, this equation states that the total energy stays constant, and that during the motion only exchanges between kinetic and potential energy occur.

In the general case, however, we will have a combination of conservative, $\boldsymbol{F}_C$, and non-conservative, $\boldsymbol{F}_{NC}$, forces. In this case, the work done by the conservative forces will be calculated using the corresponding potential function, i.e., $W_{12}^C = V_1 - V_2$, and the work done by the non-conservative forces will be path dependent and will need to be calculated using the work integral. Thus, in the general case, we will have,

$$T_1 + V_1 + \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F}_{NC} \cdot d\boldsymbol{r} = T_2 + V_2 \ .$$

The work done by non-conservative forces which oppose the motion is negative. Therefore the sum of $T_2 + V_2$ will be less than $T_1 + V_1$.

Examples of Conservative Forces

Gravity near the earth's surface

On a "flat earth", the specific gravity $\boldsymbol{g}$ points down (along the -z axis), so $\boldsymbol{F} = -mg\boldsymbol{k}$. Call $V = 0$ on the surface $z = 0$, and then

$$V(z) = -\int_0^z (-mg)\, dz, \qquad V(z) = mgz \ .$$

For the motion of a projectile, the total energy is then

$$E = \frac{1}{2}mv^2 + mgz = \text{constant} \ .$$

Since v_x and v_y remain constant, we also have $\dfrac{1}{2}mv_z^2 + mgz = \text{constant}$.

Gravity

In a central gravity field

$$\boldsymbol{F} = -G\frac{Mm}{r^2}\boldsymbol{e}_r = -\boldsymbol{\nabla}(-G\frac{Mm}{r}) \ ,$$

and so, taking $V(r \to \infty) = 0$,

$$V = -G\frac{Mm}{r} = -\frac{\mu}{r}m \ .$$

where G is the universal gravitational constant and $\mu = MG$ is the strength of the gravitational field from a central body of mass M.

Spring Force

For small displacement, the force supported by a spring is $F = -kx$. The elastic potential energy of the spring is the work done on it to deform it an amount x. Thus, we have

$$V = -\int_0^x -kx\,dx = \frac{1}{2}kx^2 \ .$$

If the deformation, either tensile or compressive, increases from x_1 to x_2 during the motion, then the change in potential energy of the spring is the difference between its final and initial values, or,

$$\Delta V = \frac{1}{2}k(x_2^2 - x_1^2) \ .$$

Gravity Potential for a Rigid Body

In this case, the potential V_i associated with particle i is simply $V_i = m_i g z_i$, where z_i is the height of particle i above some reference height. The force acting on particle i will then be $\boldsymbol{F}_i = -\boldsymbol{\nabla} V_i$. The work done on the whole body will be

$$\sum_{i=1}^{n} \int_{\boldsymbol{r}_i^1}^{\boldsymbol{r}_i^2} \boldsymbol{F}_i \cdot d\boldsymbol{r}_i = \sum_{i=1}^{n}((V_i)_1 - (V_i)_2) = \sum_{i=1}^{n} m_i g((z_i)_1 - (z_i)_2 = V_1 - V_2 \quad ,$$

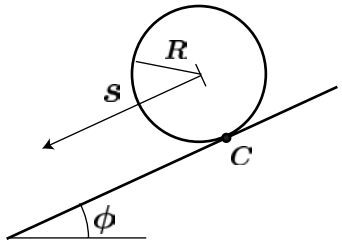
where the gravity potential for the rigid body is simply,

$$V = \sum_{i=1}^{n} m_i g z_i = m g z_G \ ,$$

where z_G is the z coordinate of the center of mass. It's obvious but worth noting that because the gravitational potential is taken about the center of mass, the inertia plays no role in determining the gravitational potential.

Example ***Cylinder on a Ramp***

We consider a homogeneous cylinder released from rest at the top of a ramp of angle ϕ, and use conservation of energy to derive an expression for the velocity of the cylinder.



Conservation of energy implies that $T+V = T_{initial}+V_{initial}$. Initially, the kinetic energy is zero, $T_{initial} = 0$. Thus, for a later time, the kinetic energy is given by

$$T = V_{initial} - V = mgs\sin\phi \ ,$$

where s is the distance traveled down the ramp. The kinetic energy is simply $T = \frac{1}{2} I_C \omega^2$, where $I_C = I_G + mR^2$ is the moment of inertia about the instantaneous center of rotation C, and ω is the angular velocity. Thus, $I_C \omega^2 = 2mgs \sin\phi$, or,

$$v^2 = \frac{2gs \sin\phi}{1 + (I_G/mR^2)} \ ,$$

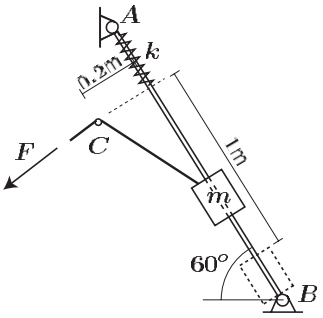
since $\omega = v/R$. For the general case of a cylinder with the center of mass at the center of the circle but an uneven mass distribution, we write $T = \frac{1}{2} m(1 + k_G^2/R^2)$, where the effect of mass distribution is captured in k_G; the smaller k_G, the more concentrated the mass about the center of the cylinder. Then

$$v^2 = \frac{2gy}{1 + k_G^2/R^2} \tag{3}$$

where $s \sin\phi$ has been replaced with the vertical distance y. This equation shows that the more the mass is concentrated towards the center of the cylinder (k_G small), a higher velocity will be reached for a given height, i.e less of the potential energy will go into rotational kinetic energy.

Example ***Principle of Work and Energy***

The $m = 30$ kg collar is released from rest at B and slides with negligible friction up the fixed rod inclined 60^o from the horizontal under the action of a constant force $\boldsymbol{F} = 450$N applied to the cable. We want to calculate the required stiffness k of the spring so that its maximum deflection equals 5cm. The position of the small pulley at C is fixed.



First, we want to calculate the work done by the cable. When the collar is at B, the length of cable between the collar and the pulley is $\sqrt{(1^2 + 0.2^2)} = 1.0198$ m. When the collar reaches its final position, the length of cable between the collar and the pulley is 0.2m. Since the force is constant, the work done by the force $\boldsymbol{F}$ on the collar is simply $W_{\text{cable}} = 450(1.0198 - 0.2) = 368.9118$ Nm. Applying the principle of work and energy between the initial position and the point of maximum spring compression (denoted by the subscript f), we have

$$T_B + V_B + W_{\text{cable}} = T_f + V_f \ .$$

Here, the kinetic energy at B and f is zero since both the initial and final velocities are zero. We can arbitrarily set the potential energy at B equal to 0. The potential energy at f will be due to gravity and to

the compression of the spring. Thus, we will have $V_f = mg(1+\delta)\sin(60^o) + k\delta^2/2$. Or,

$$368.9118 = (30)(1+0.05)\sin(60^o) + \frac{1}{2}k\,0.05^2 \qquad \rightarrow \qquad k = 81038\text{N/m}$$

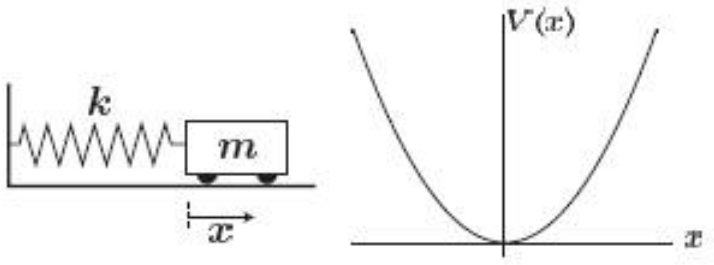
Note that when calculating the length of the cable in the final state, we have neglected the compression of the spring. This effect could easily be taken into account and the result would not differ much from the one obtained here.

Equilibrium and Stability

If all the forces acting on the body are conservative, then the potential energy can be used very effectively to determine the equilibrium positions of a system and the nature of the stability at these positions. Let us assume that all the forces acting on the system can be derived from a potential energy function, V. It is clear that if $\boldsymbol{F} = -\boldsymbol{\nabla}V = \boldsymbol{0}$ for some position, this will be a point of equilibrium in the sense that if the body is at rest (kinetic energy zero), then there will be no forces (and hence, no acceleration) to change the equilibrium, since the resultant force $\boldsymbol{F}$ is zero.

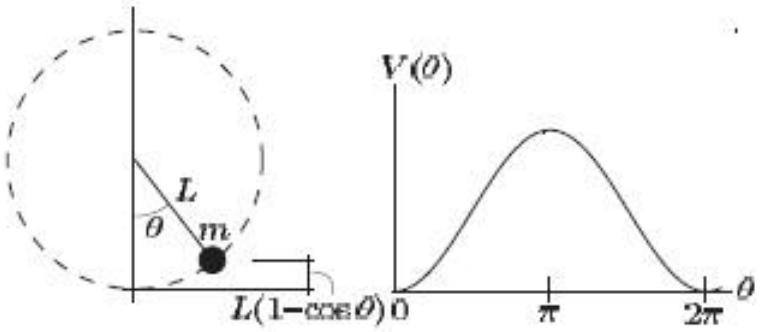
Once equilibrium has been established, the stability of the equilibrium point can be determine by examining the *shape* of the potential function. If the potential function has a *minimum* at the equilibrium point, then the equilibrium will be *stable*. This means that if the potential energy is at a minimum, there is no potential energy left that can be *traded* for kinetic energy. Analogously, if the potential energy is at a *maximum*, then the equilibrium point is *unstable*.

Let us consider a particle under the effect of a potential force. The result $\boldsymbol{F} = -\boldsymbol{\nabla}V$ is useful not only for computing the force but also for computing the stability of the motion from a diagram of the potential energy. For instance, in the case of a particle attached at the end of a spring the potential energy is $V = \frac{1}{2}kx^2$.



At a point $x > 0$, $\boldsymbol{\nabla}V = dV/dx > 0$ and so the force is negative. Similarly for $x < 0$ the force is positive. At $x = 0$, $dV/dx = 0$ and the force is zero. We see that the force is directed towards the origin no matter which way the particle is displaced and the force is only zero at the origin. The minimum of the potential energy coincides with the equilibrium position of the particle. It is clearly a stable equilibrium, since any displacement of the particle produces a force which tends to push the particle toward its resting point.

When $\nabla V = 0$ the system is in equilibrium. However, if this occurs at a maximum of V, the equilibrium is not stable, since a positive displacement produces a positive force that tends to increase the displacement. A pendulum of length L supporting as mass m is a good illustration of this.



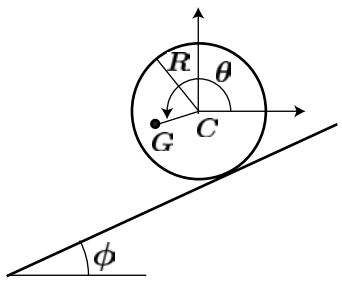
If we take the potential energy to be zero at the bottom of its swing, we see that

$$V(\theta) = mgL(1 - \cos\theta) \ .$$

The pendulum is in equilibrium for $\theta = 0$ and $\theta = \pi$. However, only $\theta = 0$ is a stable minimum since it is the only one that corresponds to a minimum.(Equilibrium requires that no force acts on the particle: $F_\theta(\theta) = \frac{dV(\theta)}{d\theta} = 0$.) When the potential is a function of just one variable (e.g. x or θ), there is a simple test that can be used to determine if the equilibrium points (i.e. points where $dV/dx = 0$) are stable or unstable. This test is based on looking at the value of the second derivative of the potential at the equilibrium point. That is if $d^2V/dx^2 > 0$ then the equilibrium point corresponds to a minimum of the potential energy and therefore, the equilibrium is *stable*. When $d^2V/dx^2 < 0$ then the equilibrium point occurs at a maximum of the potential function and the equilibrium point is *unstable*. The test only breaks down when $d^2V/dx^2 = 0$. In this case, we would need to look at higher derivatives to determine the stability of the system.

Example ***Equilibrium and Stability***

A cylinder of radius R, for which the center of gravity, G, is at a distance d from the geometric center, C, lies on a rough plane inclined at an angle ϕ.



Since gravity is the only external force acting on the cylinder that is capable of doing any work, we can examine the equilibrium and stability of the system by considering the potential energy function. We have

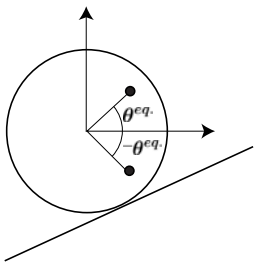
$z_C = z_{C0} - R\theta \sin\phi$, where z_{C0} is the value of z_C when $\theta = 0$. Thus, since $d = |CG|$, we have,

$$V = mgz_G = mg(z_C + d\sin\theta) = mg(z_{C0} - R\theta\sin\phi + d\sin\theta) \quad .$$

The equilibrium points are given by $\boldsymbol{\nabla} V = \mathbf{0}$, but, in this case, since the position of the system is uniquely determined by a single coordinate, e.g. θ, we can write

$$\boldsymbol{\nabla} V = \frac{dV}{d\theta}\boldsymbol{\nabla}\theta \ ,$$

which implies that, for equilibrium, $dV/d\theta = mg(-R\sin\phi + d\cos\theta) = 0$, or, $\cos\theta = (R\sin\phi)/d$. If $d < R\sin\phi$, there will be *no equilibrium* positions. On the other hand, if $d \geq R\sin\phi$, then $\theta^{eq.} = \cos^{-1}[(R\sin\phi)/d]$ is an equilibrium point. We note that if $\theta^{eq.}$ is an equilibrium point, then $-\theta^{eq.}$ is also an equilibrium point (i.e. $\cos\theta = \cos(-\theta)$).

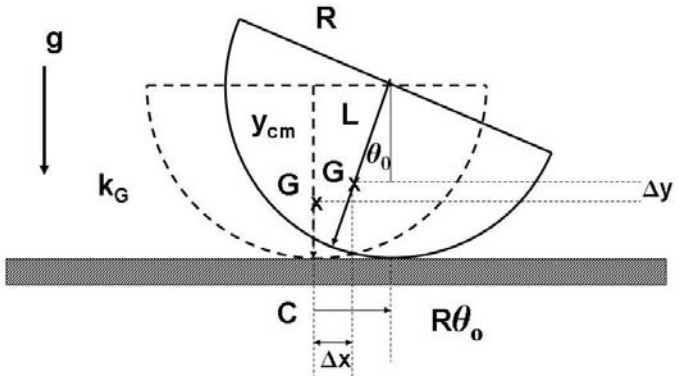


In order to study the stability of the equilibrium points, we need to determine whether the potential energy is a maximum or a minimum at these points. Since $d^2V/d\theta^2 = -mgd\sin\theta$, we have that when $\theta^{eq.} < 0$, then $d^2V/d\theta^2 > 0$ and the potential energy is a minimum at that point. Consequently, for $\theta^{eq.} < 0$, the equilibrium is stable. On the other hand, for $\theta^{eq.} > 0$, the equilibrium point is unstable.

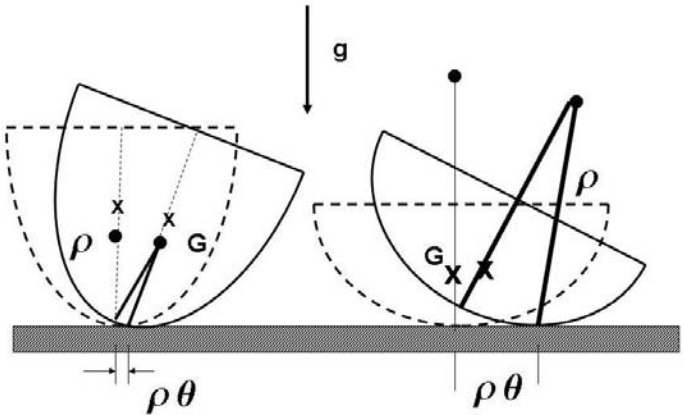
Example ***Tipped Cylinder and Ellipse***

Consider the solid semi-circle at rest on a flat plane in the presence of gravity. At rest, it is in equilibrium since the gravitational moments balance. We consider that it tips and rolls, keeping the no-slip condition satisfied. To determine the stability, we consider the change in potential energy, $V(\theta)$. Only the vertical displacement of the center of mass contributes to a change in potential. If we expand the potential $V(\theta)$ for small θ, we will get an expression $V(\theta) = A\theta^2$. (Recall that for the pendulum, $V(\theta) = mgL\theta^2/2$.) The question of stability depends upon the sign of A. If A is positive, the system is stable; if A is negative, the system is unstable.

When the cylinder tips, this motion results in a vertical displacement of the center of mass, Δy and a horizontal displacement of the center of mass Δx, where Δy and Δx can be found from the geometry. Consider the case where the center of mass is a distance L from the center of rotation of the cylinder. Then, from the figure, we see that the cylinder rolls so that the point of contact is now at $x = R\theta_0$. Then $\Delta x = R\theta_0 - L\sin\theta_0$ and $\Delta y = L(1 - \cos\theta_0)$. The vertical displacement of the center of mass is similar to that of a pendulum of length L. The tipped cylinder is stable. If the center of mass is at the center of rotation, $r = 0$, all angles θ_0 are points of neutral stability.



We now consider the two systems shown in the figure. These are simply semi-ellipses resting on a flat plane.

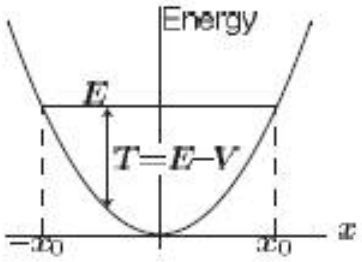


Again, the point of symmetry will be an equilibrium point since the gravitational moments will balance. But the question of stability relates to whether the center of mass moves up or down as θ increases. We have $V(\theta) = A\theta^2$, with stability for $A > 0$ and instability for $\theta < 0$. We feel instinctively, that one of these systems–the tall skinny one– might be unstable. This implies that it will not remain balanced about the equilibrium point, but will tip over. For the cylinder, the radius R played an important role. It was the distance from the point of contact to the center of curvature of the cylinder at the contact point. In this more complex example, the role of the radius R is played by the radius of curvature at the contact point ρ. Since to determine stability, we consider only small displacements the curve may be considered as a local cylinder. Referring to the figure, we see that the motion of the center of mass due to tipping of the ellipse depends on the relation between the local radius of curvature ρ and the distance of the center of mass from the center of curvature, the center of rotation. If the center of mass lies below the center of curvature, the small displacement motion will be stable, much like a pendulum. If the center of mass lies about the center of curvature, the motion will be unstable and the ellipse will initially tip over. From the figure, we see that the radius of curvature is largest for the "flat" ellipse and smallest for the tall ellipse, agreeing with our intuition about which one is more likely to tip over. However, if the center of mass of the tall ellipse is below

its center of curvature/rotation, the ellipse will be stable.

Energy Diagrams(KK)

Energy diagrams provide a useful way to study the motion of conservative one dimensional systems. In a conservative system, the total energy E is a constant; the motion transforms the form of the energy from kinetic to potential while keeping the total constant. For a given position of the system, x, the potential potential energy can be plotted, $V(x)$. The total energy of the system is constant, and is also shown in the diagram. Since the sum of the kinetic energy and the potential energy is a constant as the system moves in x, the kinetic energy $T = E - V$ is easily found by inspection. Since the kinetic energy can never be negative, the motion is constrained to regions where $V \leq E$.



Since the system is conservative, the total energy E is constant. The kinetic energy T is greatest at the origin $x = 0$. As the particle goes past the origin in either direction, it is slowed by the spring and comes to a complete rest at one of the turning points $\pm x_0$. The particle then moves to the origin increasing its kinetic energy, and the cycle is repeated. We see that in the case of a harmonic oscillator the motion is always bounded. As E increases the turning points move farther and farther off, but the particle remains bounded. Also, note that when $E = 0$ then the particle is at $x = 0$ and the particle lies at rest in equilibrium.

Example ***van der Waals Force***

The situation is different when the function V does not increase indefinitely with distance. Consider for instance the interaction between two atoms. At large separations the atoms attract each other weakly with the van der Waals force, which varies as $1/r^7$. As the atoms approach the electron clouds begin to overlap creating strong repulsive forces. The corresponding potential is given by

$$V(r) = \epsilon \left[\left(\frac{r_0}{r} \right)^{12} - 2 \left(\frac{r_0}{r} \right)^{6} \right]$$

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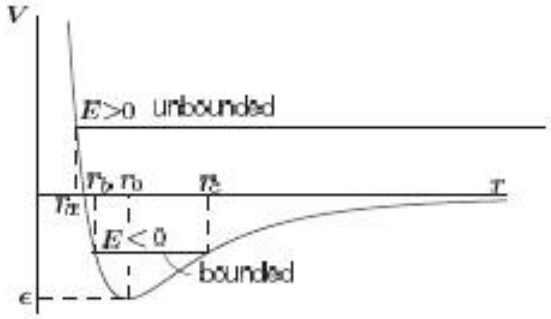
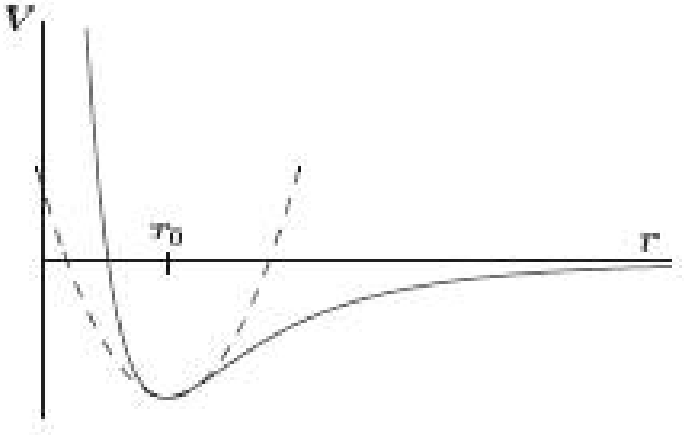


For a positive energy $E > 0$, the motion is unbounded and the atoms are free to fly apart. As the diagram shows, the distance of closest approach r_a does not change appreciably as E is increased. The situation is quite different for $E < 0$. In this case the motion is bounded for small and large separations. The atoms never approach closer then r_b and they never move apart farther than r_c. A bound system of two atoms is a molecule. If two atoms collide with positive energy they cannot form a molecule unless some means is available for losing energy to make E negative. In general a third body is necessary to carry off the excess energy.

Small Oscillations in a bound system (KK)

Every bound system oscillates as a harmonic oscillator about its equilibrium position if it is perturbed from the equilibrium position by a small amount. This can be seen by noting that the minimum of the potential energy can be generally approximated by a parabola in the neighborhood of the minimum.



If the total energy is low enough so that the motion is restricted to the region where the curve is nearly parabolic, the system will behave like a harmonic oscillator. If $V(r)$ is well behaved and has a minimum at

r_0, then we can always expand it in Taylor's series about point r_0. Thus,

$$V(r) = V(r_0) + (r - r_0)\left.\frac{dV}{dr}\right|_{r_0} + \frac{1}{2}(r - r_0)^2\left.\frac{d^2V}{dr^2}\right|_{r_0} + \dots$$

However, since at r_0, $dV/dr = 0$, for sufficiently small displacements we can truncate the series and obtain,

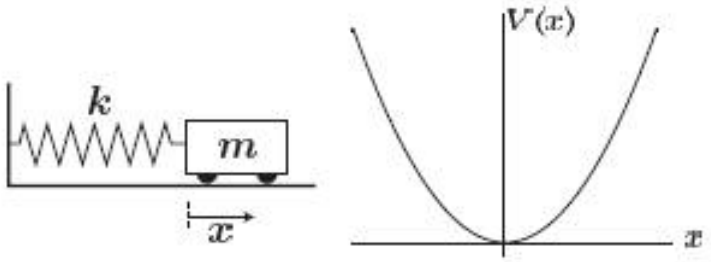
$$V(r) = V(r_0) + \frac{1}{2}(r - r_0)^2\left.\frac{d^2V}{dr^2}\right|_{r_0} .$$

$$V(r) = \text{constant} + \frac{k(r - r_0)^2}{2} .$$

We can also identify the ***effective*** spring constant as $k = d^2V/dr^2\big|_{r_0}$. These ideas can be applied to many systems, identifying the oscillatory behavior by considering the behavior of the potential function near the equilibrium point, the minimum of the potential function. (The term "effective" is used to emphasize that the stiffness in a system can be due to many effects: a spring, gravity, elasticity or a combination of these effects.) The value of the constant plays no role in the dynamics of the system.

Small Displacements of a Mass-spring System

We now consider the potential for the familiar mass-spring system , previously discussed.



For small displacements of a mass spring system, whose equilibrium position is $x = 0$, the potential function can be written

$$V(r) = \frac{kx^2}{2} .$$

where k is the spring constant. In a harmonic oscillator without damping, such as the examples discussed here, energy is conserved. As potential energy increases, kinetic energy decreases. Thus the minimum of V occurs at the maximum of T. For small amplitude motions about $x = 0$, both the displacement x and the velocity are sinusoidal in time: $x(t) = Asin(\omega t + \phi_0)$, where A and ϕ_0 are determined from the initial conditions. The kinetic energy is then given by

$$T = \frac{1}{2}mv(t)^2 = \frac{1}{2}m\omega^2(Acos(\omega t + \phi_0))^2 \tag{4}$$

where ω is the natural frequency of oscillation, that frequency which occurs as an unforced interchange between kinetic and potential energy. The oscillation occurs symmetrically about $x = 0$ the minimum of the

potential function. Since the total energy E remains constant during the oscillation

$$E = V(x(t)) + T(x(t)) = \frac{k(Asin(\omega t + \phi_0))^2}{2} + \frac{m\omega^2(Acos(\omega t + \phi_0))^2}{2} \ , \tag{5}$$

and we obtain the result $\omega = \sqrt{\frac{k}{m}}$.

We also observe that in this case the maximum value of $V(x)$ is $V_{MAX} = \frac{kA^2}{2}$; this occurs at $x = 0$. The maximum value of $T(x)$ is $T_{MAX} = \frac{m\omega^2 A^2}{2}$; this occurs at $x = A$. Equating these maximum values we again obtain $\omega = \sqrt{\frac{k}{m}}$. Any constant added to $V(x)$ plays no role in the dynamics. If $V(0) = C$, then V_{MAX} would be written $V_{MAX} = (V(x) - V(0))_{MAX}$, thus removing the constant from consideration.

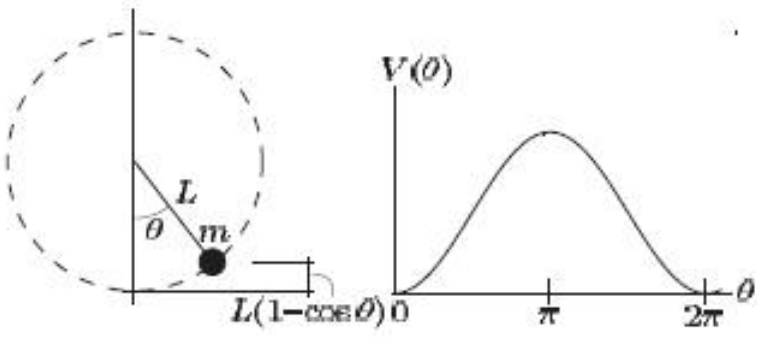
Pendulum

Earlier we derived the potential function for the pendulum as a function of the angle θ. We saw that the pendulum exhibited a range of behavior from stable oscillation about $\theta = 0$ to unstable divergence if the initial position was near $\theta = \pi$. The potential function for the pendulum was given by

$$V(\theta) = mgL(1 - \cos\theta) \ .$$

Expanding $cos\theta$ about $theta = 0$ and keeping only the terms to order θ^2 for small displacements, we obtain

$$V(\theta) = mgL(1 - \cos\theta) = mgL(1 - (1 - \frac{\theta^2}{2} + ...)) = mgL\frac{\theta^2}{2} \ .$$



For small amplitude motions about $\theta = 0$, both the displacement θ and the velocity are sinusoidal in time. As potential energy increases, kinetic energy decreases. Thus the minimum of V occurs at the maximum of T. The total energy is constant. The maximum value of kinetic energy is given by

$$T_{MAX} = \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2_{MAX} = \frac{1}{2}mL^2\omega^2\theta_0^2 \tag{6}$$

where ω is the natural frequency of oscillation and θ_0 is the maximum amplitude. The oscillation occurs symmetrically about $\theta = 0$, the minimum of the potential function. The maximum value of the potential is $V_{MAX} = mgL\frac{\theta_0^2}{2}$; this occurs at $\theta = \theta_0$ the point of maximum amplitude where the velocity is zero. Equating the maximum value of kinetic energy to the maximum value of the potential

$$T_{MAX} = \frac{1}{2}mL^2\omega^2\theta_0^2 = V_{MAX} = mgL\frac{\theta_0^2}{2} \tag{7}$$

we obtain the result $\omega = \sqrt{\frac{g}{L}}$.

Conversely if we expand the potential for small displacement near $\theta = \pi$, substituting $\theta = \pi + \alpha$, and expanding $V(\alpha)$ for small α we obtain

$$V(\alpha) = mgL(1 - cos(\pi + \alpha)) = mgL(1 + cos(\alpha)) = mgL(1 + (1 - \frac{\alpha^2}{2} + ...)) = mgL(2 - \frac{\alpha^2}{2}) \qquad (8)$$

This result completely changes the dynamics of the pendulum system. Since $F_\alpha = -\frac{dV(\alpha)}{d\alpha} = mgL\alpha$, a positive force in the direction of motion would exist. This is equivalent to a negative spring. If there was no additional restoring force, say from a spring opposing the pendulum motion, an unstable divergence would occur. (Recall that any constant adding to the value of $V(\alpha)$ is of no significance in determining the dynamics of a system; only the slope and higher order derivatives play a role.)

Oscillating Tipped Cylinder/Ellipse

We now consider how to determine the frequency of oscillation of the tipped ellipse, the cylinder being just a special case. It is obvious that the semi-circle will oscillate about its center of symmetry. The change in the gravitational potential is given by the "pendulum" formula relating the position of the center of mass L to the radius R, or for the more general case the formula relating position of the center of mass L to the radius of curvature ρ.

To determine the frequency, we need to identify T_{MAX}, the maximum value of kinetic energy. Although the moment of inertia played no role in determining the stability of the cylinder to tipping displacement, the moment of inertia will contribute to kinetic energy and thus affect the frequency of oscillation. The system has both translation and rotational kinetic energy, and both will be at their maximum values when the system moves through the point of symmetry, $\theta = 0$.

The maximum kinetic energy will be the sum of the translational and rotational kinetic energies. The maximum translational kinetic energy will be the product of the maximum velocity of the center of mass and the total mass of the cylinder; the maximum rotational kinetic energy will be the product of maximum value of the angular velocity and the moment of inertia about the center of mass. Both will reach their maximum when the system moves through $\theta = 0$, the point of minimum gravitational potential.

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ADDITIONAL READING

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