S. Widnall, J. Peraire
16.07 Dynamics
Fall 2008
Version 2.0

Lecture L12 - Work and Energy

So far we have used Newton's second law $\boldsymbol{F} = m\boldsymbol{a}$ to establish the instantaneous relation between the sum of the forces acting on a particle and the acceleration of that particle. Once the acceleration is known, the velocity (or position) is obtained by integrating the expression of the acceleration (or velocity). Newton's law and the conservation of momentum give us a vector description of the motion of a particle in three dimensions

There are two situations in which the cumulative effects of unbalanced forces acting on a particle are of interest to us. These involve:

a) forces acting along the trajectory. In this case, integration of the forces with respect to the *displacement* leads to the principle of *work* and *energy*.

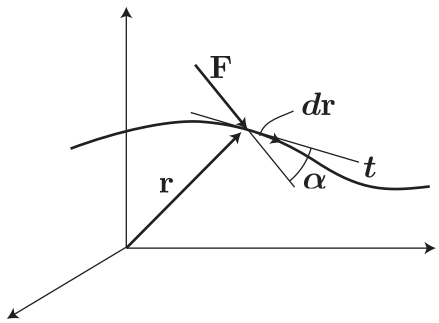
b) forces acting over a time interval. In this case, integration of the forces with respect to the *time* leads to the principle of *impulse* and *momentum* as discussed in Lecture 9.

It turns out that in many situations, these integrations can be carried beforehand to produce equations that relate the velocities at the initial and final integration points. In this way, the velocity can be obtained directly, thus making it unnecessary to solve for the acceleration. We shall see that these integrated forms of the equations of motion are very useful in the practical solution of dynamics problems.

Mechanical Work

Consider a force $\boldsymbol{F}$ acting on a particle that moves along a path. Let $\boldsymbol{r}$ be the position of the particle measured relative to the origin O. The work done by the force $\boldsymbol{F}$ when the particle moves an infinitesimal amount $d\boldsymbol{r}$ is defined as

$$dW = \boldsymbol{F} \cdot d\boldsymbol{r} \ . \qquad (1)$$



That is, the work done by the force $\boldsymbol{F}$, over an infinitesimal displacement $d\boldsymbol{r}$, is the *scalar* product of $\boldsymbol{F}$ and $d\boldsymbol{r}$. It follows that the work is a *scalar* quantity. Using the definition of the scalar product, we have that $dW = \boldsymbol{F} \cdot d\boldsymbol{r} = F ds \cos\alpha$, where ds is the modulus of $d\boldsymbol{r}$, and α is the angle between $\boldsymbol{F}$ and $d\boldsymbol{r}$. Since $d\boldsymbol{r}$ is parallel to the tangent vector to the path, $\boldsymbol{e}_t$, (i.e. $d\boldsymbol{r} = ds\, \boldsymbol{e}_t$), we have that $\boldsymbol{F} \cdot \boldsymbol{e}_t = F_t$. Thus,

$$dW = F_t \, ds \ , \tag{2}$$

which implies that only the tangential component of the force "does" work.

During a finite increment in which the particle moves from position $\boldsymbol{r}_1$ to position $\boldsymbol{r}_2$, the total work done by $\boldsymbol{F}$ is

$$W_{12} = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{s_1}^{s_2} F_t \, ds \ . \tag{3}$$

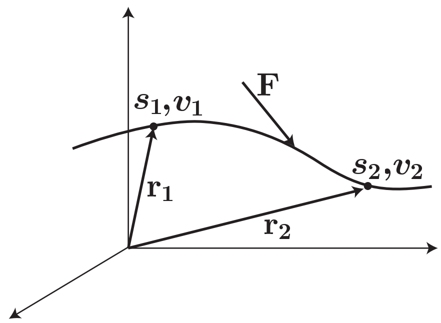
Here, s_1 and s_2 are the path coordinates corresponding to $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$.

Note ***Units of Work***

In the international system, SI, the unit of work is the *Joule* (J). We have that 1 J = 1 N · m. In the English system the unit of work is the ft-lb. We note that the units of work and moment are the same. It is customary to use ft-lb for work and lb-ft for moments to avoid confusion.

Principle of Work and Energy

We now a consider a particle moving along its path from point $\boldsymbol{r}_1$ to point $\boldsymbol{r}_2$. The path coordinates at points 1 and 2 are s_1 and s_2, and the corresponding velocity magnitudes v_1 and v_2.



If we start from (3) and use Newton's second law ($\boldsymbol{F} = m\boldsymbol{a}$) to express $F_t = ma_t$, we have

$$W_{12} = \int_{s_1}^{s_2} F_t \, ds = \int_{s_1}^{s_2} ma_t \, ds = \int_{v_1}^{v_2} m \frac{dv}{ds}\frac{ds}{dt} \, ds = \int_{v_1}^{v_2} mv \, dv = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \ . \qquad (4)$$

Here, we have used the relationship $a_t \, ds = v \, dv$, which can be easily derived from $a_t = \dot{v}$ and $v = \dot{s}$ (see lecture D4).

Defining the *kinetic energy*[1], as

$$T = \frac{1}{2}mv^2 \ ,$$

we have that,

$$W_{12} = T_2 - T_1 \qquad \text{or} \qquad T_1 + W_{12} = T_2 \ . \qquad (5)$$

The above relationship is known as the *principle of work and energy*, and states that the mechanical work done on a particle is equal to the change in the kinetic energy of the particle.

External Forces

Since the body is rigid and the internal forces act in equal and opposite directions, only the external forces applied to the rigid body are capable of doing any work. Thus, the total work done on the body will be

$$\sum_{i=1}^{n} (W_i)_{1-2} = \sum_{i=1}^{n} \int_{(\boldsymbol{r}_i)_1}^{(\boldsymbol{r}_i)_2} \boldsymbol{F}_i \cdot d\boldsymbol{r} \quad ,$$

where $\boldsymbol{F}_i$ is the sum of all the **external** forces acting on particle i.

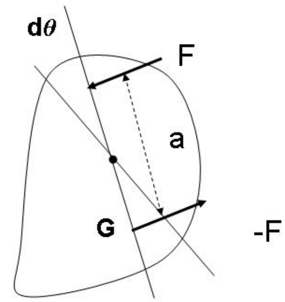
Work done by couples

If the sum of the external forces acting on the rigid body is zero, it is still possible to have non-zero work. Consider, for instance, a moment $M = Fa$ acting on a rigid body. If the body undergoes a pure translation, it is clear that all the points in the body experience the same displacement,

[1]The use of T to denote the kinetic energy, instead of K, is customary in dynamics textbooks

and, hence, the total work done by a couple is zero. On the other hand, if the body experiences a rotation $d\theta$, then the work done by the couple is

$$dW = F\,\frac{a}{2}d\theta + F\,\frac{a}{2}d\theta = F\,ad\theta = Md\theta\ .$$



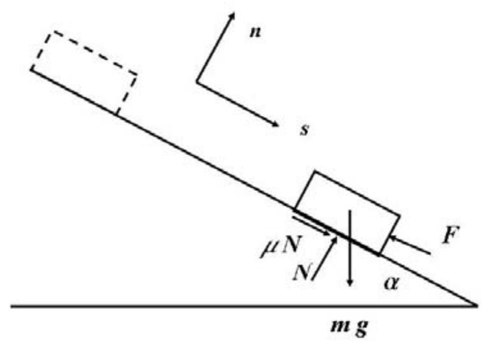
If M is constant, the work is simply $W_{1-2} = M(\theta^2 - \theta^1)$. In other words, the couples do work which results in the kinetic energy of rotation.

External and Internal Forces

In a typical dynamical system, the force $\boldsymbol{F}$ is composed of two terms: an external force $\boldsymbol{F}_E$ and an internal force $\boldsymbol{F}_I$. The external force results from an external actor applying an arbitrary force–of his choice– to the system. The external force does work and changes the energy of the system at the whim of the actor. When the external force is removed, the system may oscillate or otherwise move subject to internal forces. Internal forces are of two types: conservative internal forces such as gravity which conserve energy but for example can transform kinetic into potential energy and vice versa; and friction which acts internally to dissipate energy from the system. Initial conditions are often set by applied external forces to the system, such as doing work by moving a pendulum through an initial angle θ_0. When they are removed, the system oscillates perhaps conserving energy if friction is absent. We will consider conservative forces shortly.

Example — ***Block on an incline***

Consider a block resting on an incline at position 1 in the presence of gravity. The gravitational force acting in the vertical direction is mg. The block is supported on the plane by a normal force $F_N = mgsin\alpha$. We desire to push the block up to position 2. To do this, we must apply an **external** force to overcome the component of gravity (internal force) along the plane plus the friction force (internal force) acting to oppose the motion. In this process, the normal force does no work.



The total tangential force acting on the block is then

$$F_T = F_E + F_I \tag{6}$$

with

$$F_E = -mgsin\alpha - \mu mgcos\alpha \tag{7}$$

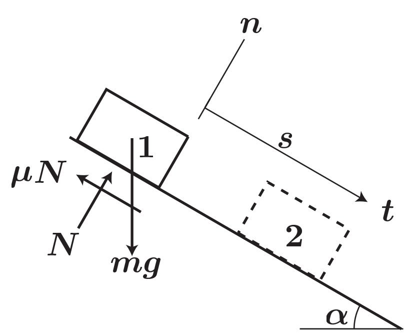
$$F_I = +mgsin\alpha + \mu mgcos\alpha \tag{8}$$

for a total force, internal plus external, of zero. Under these assumptions, the block arrives at position 2 with zero velocity. Therefore, by equation (5), no work is done. At first, this does not agree with our intuition. We certainly felt we did work in pushing the block up the plane. But the work done by the friction force and gravitational force exactly canceled this work. We also feel that having raised the block to a higher position, there is some inherent "gain" in energy which could be collected in the future. This is true! This result –that the total force is zero resulting in "no work" and no kinetic energy being gained–is due to the ***external*** application of force. These external forces exactly canceled the internal forces of gravity and friction, driving the total force to zero and resulting in no total work In other words, we became an actor instead of an observer. These issues will be considered further when we consider conservative forces and potentials

---

Example **Block sliding down an incline**

Having applied an external force F_E to move the block to a higher position on the ramps where it rests with no kinetic energy, we now become an observer and release it with only the internal forces of gravity and friction acting. The coefficient of kinetic friction between the surface of the ramp and the block is μ. We want to determine the velocity of the block as a function of the distance traveled on the ramp, s.



The forces on the block are: the weight, mg, the normal force, N, and, the friction force, μN. We have that $F_n = ma_n$ and $F_t = ma_t$. Since $F_n = N - mg\cos\alpha$ and $a_n = 0$, we have $N = mg\cos\alpha$. Thus, $F_t = mg\sin\alpha - \mu N = mg\sin\alpha - \mu mg\cos\alpha$, which is constant. If we apply the principle of work and energy between the position (1), when the block is at rest at the top of the ramp, and the position (2), when the block has traveled a distance s, we have $T_1 = 0$, $T_2 = (mv^2)/2$, and the work done by F_t is simply $W_{12} = F_t s$. Thus,

$$T_1 + W_{12} = T_2 \ , \qquad \text{or,} \qquad mg(\sin\alpha - \mu\cos\alpha)s = \frac{1}{2}mv^2 \ .$$

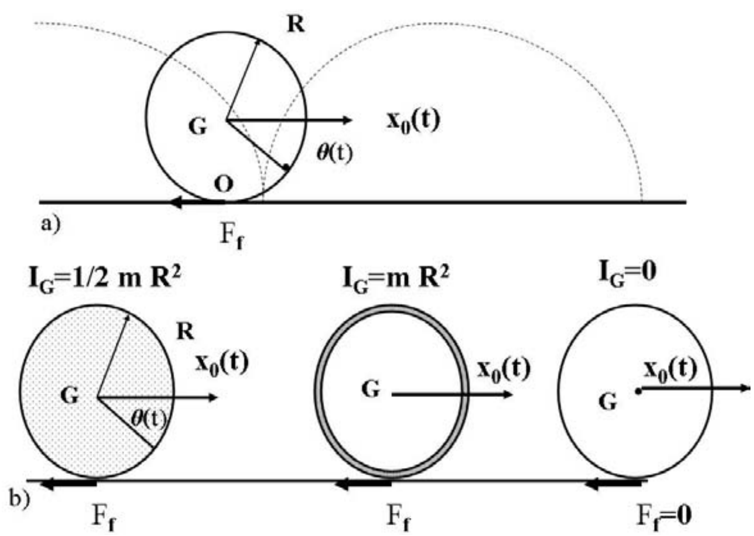
From which we obtain, for the velocity,

$$v = \sqrt{2g(\sin\alpha - \mu\cos\alpha)s} \ .$$

We make two observations: first, the normal force, N, does no work since it is, at all times, perpendicular to the path, and second, we have obtained the velocity of the block directly without having to carry out any integrations. Note that an alternative, longer approach would have been to directly use $\boldsymbol{F} = m\boldsymbol{a}$, and integrate the corresponding expression for the acceleration.

Rolling Cylinder, Friction Forces, Work

The cylinder rolling on a flat plane is a very basic configuration in dynamics. As noted in Lecture 2, it is a single degree of freedom system with a definite relationship between the position of the center of the cylinder, $x_0(t)$ and the rotation angle $\theta(t)$: $\theta(t) = -x_0(t)/R$. The kinematics of the rolling cylinder are shown in a). Consider a mass point at the edge of the cylinder. The dashed curve shows the path taken by this mass point. Of significance is the behavior near the plane. The point O is an instantaneous center of rotation; the tangential velocity of the cylinder is zero about this point. The acceleration of the mass point is not zero, nor is its vertical velocity.



The dynamics of the rolling cylinder is shown in b). If the cylinder moves with constant velocity, nothing more need be said. However, if $\ddot{x}_0$ is not zero, then $\dot{\omega} = \alpha$, the angular acceleration will be nonzero. This will require a moment about the center of mass, which in the simplest configuration sketched, can only come from friction with the plane. $F_f R = I_G \alpha$. Since the point in contact with the plane is an instantaneous center of rotation–does not move–this friction force does no work. Also shown in b) are a variety of configurations of rolling cylinders. The solid cylinder has a moment of inertia of $I_G = 1/2mR^2$; the cylinder whose mass is concentrated in the rim, has a moment of inertia $I_G = mr^2$; the cylinder whose mass point is concentrated in the center has a moment of inertia of zero. Therefore, no moment is required to change its angular velocity and it behaves like a mass point moving on a frictionless surface. The collision of two mass points can easily be realized by the collision of two such rolling cylinders.

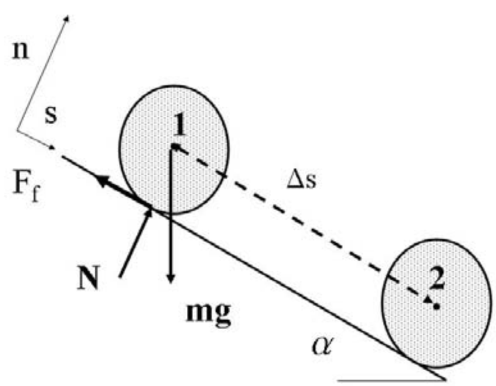
Note ***Cylinder rolling down a ramp***

In parallel with our discussion of a block sliding down a ramp in the presence of friction, we now consider a cylinder rolling down a ramp in the presence of friction. We assume that friction forces are large enough to keep the kinematic relationship between the velocity of the cylinder and the angular velocity of the cylinder intact. The cylinder is located at an initial position **1** as shown, and is at rest. It is released and rolls down to position **2** where we observe it. The forces acting on the cylinder are gravity, the normal force **N**, and the friction force $\mathbf{F}_f$. Although the friction force is necessary to keep the kinematic relationship intact, **it does no work**; as before, the normal force does no work. Therefore, the only work is done by gravity. Thus we can write

$$T_1 + W_{12} = T_2 \tag{9}$$

The initial kinetic energy is zero. The work done by gravity is $mg \sin \alpha \Delta s$; the final kinetic energy, which includes both the kinetic energy due to translation and the kinetic energy due to rotation is $T_2 = 1/2mv^2 + 1/2I_G(v/R)^2$. Therefore, the final velocity is

$$v = \sqrt{\frac{2 * g \sin \alpha \Delta s}{1 + I_G/R^2}} \tag{10}$$



Note ***Alternative expressions for*** dW

We have seen in expression (2) that a convenient set of coordinates to express dW are the tangential-normal-binormal coordinates. Alternative expressions can be derived for other coordinate systems. For instance, we can express $dW = \boldsymbol{F} \cdot d\boldsymbol{r}$ in:

cartesian coordinates,

$$dW = F_x\, dx + F_y\, dy + F_z\, dz \ ,$$

cylindrical (polar) coordinates,

$$dW = F_r\, dr + F_\theta\, r d\theta + F_z\, dz \ ,$$

or spherical coordinates,

$$dW = F_r\, dr + F_\theta\, r \cos\phi\, d\theta + F_\phi\, r d\phi \ .$$

As an illustration, let's calculate the work done by a constant internal force, such as that due to gravity. The force on a particle of mass m is given by $\boldsymbol{F} = -mg\boldsymbol{k}$. When the particle moves from position $\boldsymbol{r}_1 = x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}$ to position $\boldsymbol{r}_2 = x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k}$, work is done, and the work may be written as

$$W_{12} = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{z_1}^{z_2} -mg\, dz = -mg(z_2 - z_1) \ .$$

Power

In many situations it is useful to consider the rate at which a device can deliver work. The work per unit time is called the *power*, P. Thus,

$$P = \frac{dW}{dt} = \boldsymbol{F} \cdot \frac{d\boldsymbol{r}}{dt} = \boldsymbol{F} \cdot \boldsymbol{v} \ .$$

The unit of power in the SI system is the *Watt* (W). We have that 1 W = 1 J /s. In the English system the unit of power is the ft-lb/s. A common unit of power is also the *horse power* (hp), which is equivalent to 550 ft-lb/s, or 746 W.

Note ***Efficiency***

The ratio of the power delivered *out of* a system, P_{out}, to the power delivered *in to* the system, P_{in}, is called the *efficiency*, e, of the system.

$$e = \frac{P_{out}}{P_{in}} \ .$$

This definition assumes that the energy into and out of the system flows continuously and is not retained within the system. The efficiency of any real machine is always less than unity since there is always some mechanical energy dissipated as heat due to friction forces.

ADDITIONAL READING

- J. B. Marion, S. T. Thornton *Classical Dynamics of Particles and Systems*, Harcourt Brace, New York, Section 2.5
- J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition