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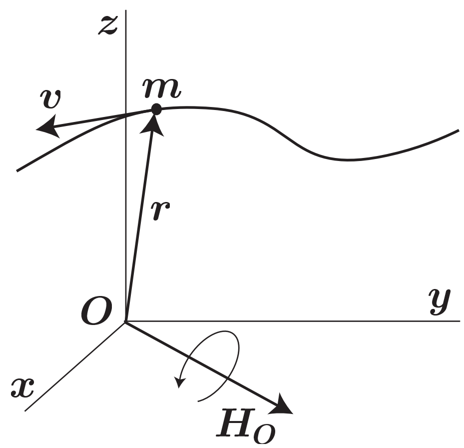
Lecture L10 - Angular Impulse and Momentum for a Particle

In addition to the equations of linear impulse and momentum considered in the previous lecture, there is a parallel set of equations that relate the angular impulse and momentum.

Angular Momentum

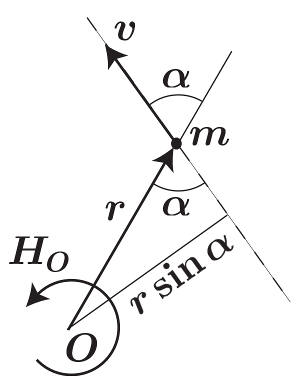
We consider a particle of mass m, with velocity $\boldsymbol{v}$, moving under the influence of a force $\boldsymbol{F}$. The angular momentum about point O is defined as the "moment" of the particle's linear momentum, $\boldsymbol{L}$, about O.



Thus, the particle's angular momentum is given by,

$$\boldsymbol{H}_O = \boldsymbol{r} \times m\boldsymbol{v} = \boldsymbol{r} \times \boldsymbol{L} \ . \tag{1}$$

The units for the angular momentum are kg·m^2/s in the SI system, and slug·ft^2/s in the English system.



It is clear from its definition that the angular momentum is a vector which is perpendicular to the plane defined by $\boldsymbol{r}$ and $\boldsymbol{v}$. Thus, on some occasions it may be more convenient to determine the direction of $\boldsymbol{H}_O$ from the right hand rule, and its modulus directly from the definition of the vector product,

$$H_O = mvr\sin\alpha \ ,$$

where α is the angle between $\boldsymbol{r}$ and $\boldsymbol{v}$.

In other situations, it may be convenient to directly calculate the angular momentum in component form. For instance, using a right handed cartesian coordinate system, the components of the angular momentum are calculated as

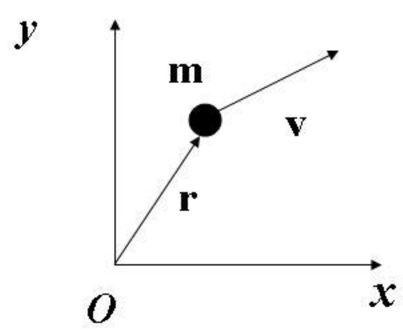
$$\boldsymbol{H}_O = H_x\boldsymbol{i} + H_y\boldsymbol{j} + H_z\boldsymbol{k} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & z \\ mv_x & mv_y & mv_z \end{vmatrix} = m(v_z y - v_y z)\boldsymbol{i} + m(v_x z - v_z x)\boldsymbol{j} + m(v_y x - v_x y)\boldsymbol{k} \ .$$

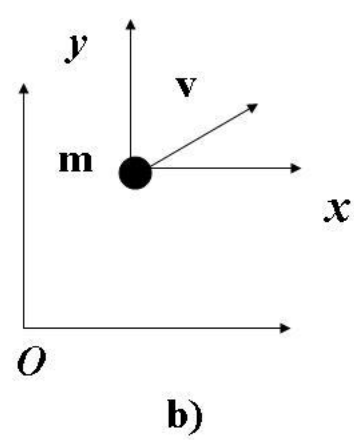
Similarly, in cylindrical coordinates we have

$$\boldsymbol{H}_O = H_r\boldsymbol{e}_r + H_\theta\boldsymbol{e}_\theta + H_z\boldsymbol{k} = \begin{vmatrix} \boldsymbol{e}_r & \boldsymbol{e}_\theta & \boldsymbol{k} \\ r & 0 & z \\ mv_r & mv_\theta & mv_z \end{vmatrix} = -mv_\theta z\boldsymbol{e}_r + m(v_r z - v_z r)\boldsymbol{e}_\theta + mv_\theta r\boldsymbol{k} \ .$$

Rate of Change of Angular Momentum

We now want to examine how the angular momentum changes with time. We examine this in two different coordinate systems: system a) is about a fixed point O; system b) is about the center of mass of the particle. Of course system b) is rather trivial for a point mass, but its later extensions to finite bodies will be extremely important. Even at this trivial level, we will obtain an important result.





About a Fixed Point O

The angular momentum about the fixed point O is

$$H_O = \boldsymbol{r} \times m\boldsymbol{v} \tag{2}$$

Taking a time derivative of this expression , we have

$$\dot{\boldsymbol{H}}_O = \dot{\boldsymbol{r}} \times m\boldsymbol{v} + \boldsymbol{r} \times m\dot{\boldsymbol{v}} \ .$$

Here, we have assumed that m is constant. If O is a **fixed** point, then $\dot{\boldsymbol{r}} = \boldsymbol{v}$ and $\dot{\boldsymbol{r}} \times m\boldsymbol{v} = 0$. Thus, we end up with,

$$\dot{\boldsymbol{H}}_O = \boldsymbol{r} \times m\dot{\boldsymbol{v}} = \boldsymbol{r} \times m\boldsymbol{a} \ .$$

Applying Newton's second law to the right hand side of the above equation, we have that $\boldsymbol{r} \times m\boldsymbol{a} = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{M}_O$, where $\boldsymbol{M}_O$ is the moment of the force $\boldsymbol{F}$ about point O. The equation expressing the rate of change of angular momentum is then written as

$$\boldsymbol{M}_O = \dot{\boldsymbol{H}}_O \ . \tag{3}$$

We note that this expression is valid whenever point O is fixed. The above equation is analogous to the equation derived in the previous lecture expressing the rate of change of linear momentum. It states that the rate of change of linear momentum about a fixed point O is equal to the moment about O due to the resultant force acting on the particle. Since this is a vector equation, it must be satisfied for each component independently. Thus, if the force acting on a particle is such that the component of its moment along a given direction is zero, then the component of the angular momentum along this direction will remain constant. This equation is a direct consequence of Newton's law. It will not give us more information about the momentum of a particle, but a clever choice of coordinates may make angular momentum easier to apply in any given case.

About the Center of Mass

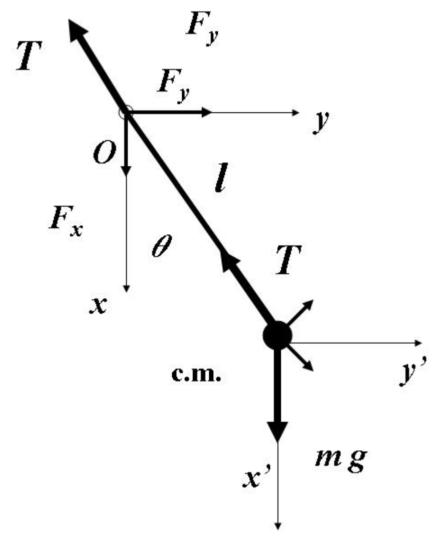
For a particle, the angular momentum is zero. We examine carefully the expression for the rate of change of angular momentum.

$$\dot{\boldsymbol{H}}_O = \dot{\boldsymbol{r}} \times m\boldsymbol{v} + \boldsymbol{r} \times m\dot{\boldsymbol{v}} \ .$$

Since the coordinate system moves with the particle, both $\dot{\boldsymbol{r}}$ and $\dot{\boldsymbol{v}}$ are zero. This is true even if the coordinate system is not inertial, in contrast to the application of Newton's Law for linear momentum. Therefore, for a mass point the rate of change of angular momentum is zero in a coordinate system moving with the particle. This implies that no moments can be applied to a mass point in a coordinate system moving with the point. This result will later be extended to bodies of finite size; we can apply Equation 3 to a body of finite size even in an accelerating coordinate system if the origin of our coordinate system is the center of mass.

Example ***Pendulum***

Here, we consider the simple pendulum problem. However, we assume that the point mass m is suspended from a rod attached at a pivot that could support a side force. Therefore we allow the direction of the force to be unknown. We first apply conservation of angular momentum in a coordinate system moving with the point mass, the x', y' system. The result that the angular momentum in this coordinate system is zero, gives us an immediate result that no external torques can act on the particle. Gravity acts at the particle and therefore produces no moment. Therefore we conclude that the force in the rod must point directly to the mass, along the rod itself. In other words the rod acts as a string supporting a tension T. We will later see that if the pendulum has a finite moment of inertia about the center of mass, this result no longer applies. We now examine the pendulum in a coordinate system fixed at the point O and re-derive the pendulum equation using equation (3).



There are two forces acting on the suspended mass: the string tension and the weight. By our earlier argument, the tension, $\boldsymbol{T}$, is parallel to the position vector $\boldsymbol{r}$ and therefore its moment about O is zero. On the other hand, the weight creates a moment about O which is $\boldsymbol{M}_O = -lmg \sin\theta \boldsymbol{k}$.
The angular momentum is given by

$$\boldsymbol{H}_O = \boldsymbol{r} \times m\boldsymbol{v} = l\boldsymbol{e}_r \times ml\dot{\theta}\boldsymbol{e}_\theta = ml^2\dot{\theta}\ \boldsymbol{e}_r \times \boldsymbol{e}_\theta = ml^2\dot{\theta}\boldsymbol{k}.$$

Therefore, the z component of equation (3) gives

$$ml^2\ddot{\theta} = -lmg \sin\theta\ ,$$

or,

$$\ddot{\theta} + \frac{g}{l} \sin\theta = 0\ ,$$

which is precisely the same equation as the one derived in lecture L5 using Newton's law. The derivation using angular momentum is more compact.

Principle of Angular Impulse and Momentum

Equation (3) gives us the instantaneous relation between the moment and the time rate of change of angular momentum. Imagine now that the force considered acts on a particle between time t_1 and time t_2. Equation (3) can then be integrated in time to obtain

$$\int_{t_1}^{t_2} \boldsymbol{M}_O \, dt = \int_{t_1}^{t_2} \dot{\boldsymbol{H}}_O \, dt = (\boldsymbol{H}_O)_2 - (\boldsymbol{H}_O)_1 = \Delta \boldsymbol{H}_O \ . \tag{4}$$

Here, $(\boldsymbol{H}_O)_1 = \boldsymbol{H}_O(t_1)$ and $(\boldsymbol{H}_O)_2 = \boldsymbol{H}_O(t_2)$. The term

$$\int_{t_1}^{t_2} \boldsymbol{M}_O \, dt \ ,$$

is called the *angular impulse*. Thus, the angular impulse on a particle is equal to the angular momentum change.

Equation (4) is particularly useful when we are dealing with impulsive forces. In such cases, it is often possible to calculate the integrated effect of a force on a particle without knowing in detail the actual value of the force as a function of time.

Conservation of Angular Momentum

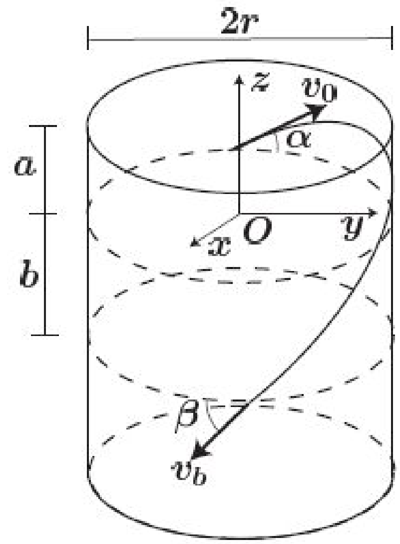
We see from equation (1) that if the moment of the resultant force on a particle is zero during an interval of time, then its angular momentum $\boldsymbol{H}_O$ must remain constant.

Consider now two particles m_1 and m_2 which interact during an interval of time. Assume that interaction forces between them are the only unbalanced forces on the particles that have a non-zero moment about a fixed point O. Let $\boldsymbol{F}$ be the interaction force that particle m_2 exerts on particle m_1. Then, according to Newton's third law, the interaction force that particle m_1 exerts on particle m_2 will be $-\boldsymbol{F}$. Using expression (4), we will have that $\Delta(\boldsymbol{H}_O)_1 = -\Delta(\boldsymbol{H}_O)_2$, or $\Delta \boldsymbol{H}_O = \Delta(\boldsymbol{H}_O)_1 + \Delta(\boldsymbol{H}_O)_2 = \boldsymbol{0}$. That is, the changes in angular momentum of particles m_1 and m_2 are equal in magnitude and of opposite sign, and the total angular momentum change equals zero. Recall that this is true only if the unbalanced forces, those with non-zero moment about O, are the interaction forces between the particles. The more general situation in which external forces can be present will be considered in future lectures.

We note that the above argument is also valid in a componentwise sense. That is, when two particles interact and there are no external unbalanced moments along a given direction, then the total angular momentum change along that direction must be zero.

Example | ***Ball on a cylinder***

A particle of mass m is released on the smooth inside wall of an open cylindrical surface with a velocity $\boldsymbol{v}_0$ that makes an angle α with the horizontal tangent. The gravity acceleration is pointing downwards. We want to obtain : i) an expression for the largest magnitude of $\boldsymbol{v}_0$ that will prevent the particle from leaving the cylinder through the top end, and ii) an expression for the angle β that the velocity vector will form with the horizontal tangent, as a function of b.



The only forces on the particle are gravity and the normal force from the cylinder surface. The moment of these forces about O (or, in fact, about any point on the axis of the cylinder) always has a zero component in the z direction. That is, $(M_O)_z = 0$. To see that, we notice that for any point on the surface of the cylinder, $\boldsymbol{r}$ and $\boldsymbol{F}$ are always contained in a vertical plane that contains the z axis. Therefore, the moment must be normal to that plane. Since the moment has zero component in the z direction, $(H_O)_z$ will be constant. Thus, we have that

$$(H_O)_z = rmv_0 \cos\alpha = \text{constant} .$$

For part i), we consider the trajectory for which the velocity is horizontal when $z = a$ and let $(\boldsymbol{v}_0)_{\text{limit}}$ be the initial velocity that corresponds to this trajectory. It is clear that for any trajectory for which $\boldsymbol{v}_0$ has a larger magnitude than $(\boldsymbol{v}_0)_{\text{limit}}$, the particle will leave the cylinder through the top end. Thus, for the limit trajectory we have, from conservation of energy

$$\frac{1}{2}m(v_0)^2_{\text{limit}} = \frac{1}{2}mv_a^2 + mga ,$$

and from conservation of angular momentum

$$(H_O)_z = rm(v_0)_{\text{limit}} \cos\alpha = rmv_a .$$

Here, v_a is the magnitude of the velocity for the limit trajectory when $z = a$. Eliminating v_a from these equations we finally arrive at,

$$(v_0)_{\text{limit}} = \sqrt{\frac{2ga}{1 - \cos^2\alpha}} .$$

Therefore, for $v_0 \leq (v_0)_{\text{limit}}$ the mass will not leave the cylinder through the top end.

For part ii), we also consider conservation of energy

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_b^2 - mgb$$

and conservation of angular momentum,

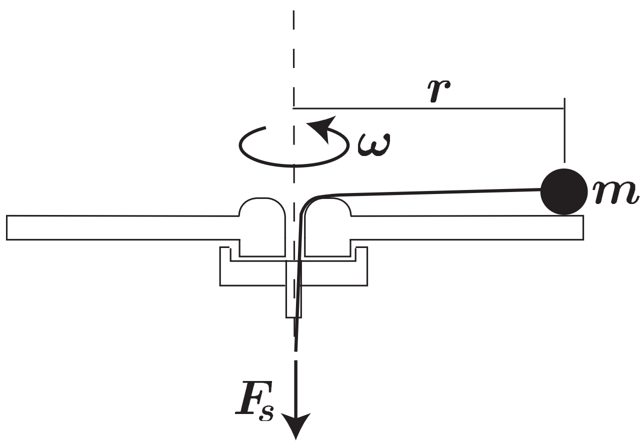
$$rmv_0 \cos\alpha = rmv_b \cos\beta \ .$$

Eliminating, v_b from these two expressions we obtain,

$$\beta = \cos^{-1}\left(\frac{\cos\alpha}{\sqrt{1+2gb/v_0^2}}\right) \ .$$

Example ***Spinning Mass***

A small particle of mass m and its restraining cord are spinning with an angular velocity ω on the horizontal surface of a smooth disk, shown in section. As the force $\boldsymbol{F}_s$ is slowly increased, r decreases and ω changes. Initially, the mass is spinning with ω_0 and r_0. Determine : i) an expression for ω as a function of r, and ii) the work done on the particle by $\boldsymbol{F}_s$ between r_0 and an arbitrary r. Verify the principle of work and energy.



The component of the moment of the forces acting on the particle is zero along the spinning axis. Therefore, the vertical component of the angular momentum will be constant. For i), we have

$$mr_0v_0 = mrv, \quad v_0 = \omega_0 r_0, \ v = \omega r \quad \rightarrow \quad \omega = \frac{r_0^2\omega_0}{r^2} \ .$$

For ii), we first calculate the force on the string

$$F_s = -m\frac{v^2}{r} = -m\frac{r^2\omega^2}{r} = -m\frac{r_0^4\omega_0^2}{r^3} \ .$$

The work done by F_s, will be

$$W = \int_{r_0}^{r} F_s dr = -mr_0^4\omega_0^2\int_{r_0}^{r}\frac{dr}{r^3} = mr_0^4\omega_0^2\frac{1}{2}\left(\frac{1}{r^2} - \frac{1}{r_0^2}\right) \ .$$

The energy balance implies that

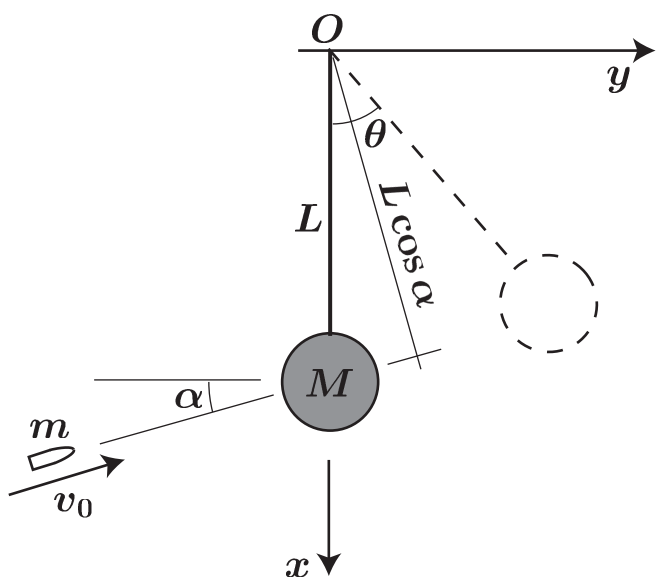
$$T_0 + W = T \ .$$

This expression can be directly verified since,

$$\underbrace{\frac{1}{2}m(\omega_0 r_0)^2}_{T_0} + \underbrace{\frac{1}{2}m(\omega_0 r_0)^2\left(\frac{r_0^2}{r^2} - 1\right)}_{W} = \underbrace{\frac{1}{2}m(\omega r)^2}_{T} \ .$$

Example ***Ballistic Pendulum***

We consider a pendulum consisting of a mass, M, suspended by a *rigid* rod of length L. The pendulum is initially at rest and the mass of the rod can be neglected. A bullet of mass m and velocity $\boldsymbol{v}_0$ impacts M and stays embedded in it. We want to find out the angle $\theta_{\max}$ reached by the pendulum. The angle that the velocity vector $\boldsymbol{v}_0$ forms with the horizontal is α.



Because the rod is assumed to be rigid, we can expect that when the bullet impacts the mass, there will be an impulsive reaction that the rod will exert on the bullet. If we use the principle of linear impulse and momentum, it will be necessary to solve for this impulsive force. An alternative approach that simplifies the problem considerably is to use the principle of angular impulse and momentum. We consider the angular momentum about point O of the particles m and M just before and after the impact. The only external forces acting on the two particles are gravity and the reaction from the rod. It turns out that gravity is not an impulsive force and therefore its effect on the total angular impulse, over a very short time interval, can be safely neglected (it turns out that in this case, the moment about O of the gravity forces at the time of impact is also zero). On the other hand, we can expect the reaction from the rod to be large. However, the moment about O of this reaction is zero, since it is directed in the direction of the rod. Therefore, we have that during impact, the z component of the angular momentum is conserved.

The angular momentum before impact will be,

$$[(H_O)_z]_1 = L\cos\alpha\; mv_0\ .$$

After impact, the velocity $\boldsymbol{v}_1$ has to be horizontal. Thus, the angular momentum will be

$$[(H_O)_z]_2 = L(M+m)v_1\ .$$

Equating these two expressions we get,

$$v_1 = \frac{m}{M+m} v_0 \cos\alpha\ .$$

After impact, the system is conservative, and the maximum height can be easily obtained from conservation of energy,

$$\frac{1}{2}(M+m)v_1^2 = (M+m)gL(1-\cos\theta_{\max})\ .$$

Thus,

$$\theta_{\max} = \cos^{-1}\left(1 - \left(\frac{m}{(M+m)}\right)^2 \frac{v_0^2\cos^2\alpha}{2gL}\right)\ .$$

ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition

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