Lecture L6 - Intrinsic Coordinates

In lecture L4, we introduced the position, velocity and acceleration vectors and referred them to a fixed cartesian coordinate system . Then we showed how they could be expressed in polar coordinates. While it is clear that the choice of coordinate system does not affect the final answer, we shall see that, in practical problems, the choice of a specific system may simplify the calculations and/or improve the understanding considerably. In many problems, it is useful to use a coordinate system that aligns with both the velocity vector of the particle—which is of course tangent to the particle's trajectory— and the normal to this trajectory, forming a pair of orthogonal unit vectors. The unit vectors aligned with these two directions also define a third direction, call the binormal which is normal to both the velocity vector and the normal vector. This can be obtained from these two unit vectors by taking their cross product. Such a coordinate system is called an ***Intrinsic Coordinate System***.

## Intrinsic coordinates: Tangential, Normal and Binormal components.

We follow the motion of a point using a vector $\boldsymbol{r}(t)$ whose position along a known curved path is given by the ***scalar function*** $s(t)$ where $s(t)$ is the arc length along the curve. We obtain the velocity $\boldsymbol{v}$ from the time rate of change of the vector $\boldsymbol{r}(t)$ following the particle

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}}{ds}\frac{ds}{dt} = \dot{s}\frac{d\boldsymbol{r}}{ds} \tag{1}$$

We identify ***the scalar*** $\dot{s}$ as the magnitude of the velocity v and $\frac{d\boldsymbol{r}}{ds}$ as the unit vector tangent to the curve at the point $s(t)$. Therefore we have

$$\boldsymbol{v} = v\boldsymbol{e}_t \ , \tag{2}$$

where $\boldsymbol{r}(t)$ is the position vector, $v = \dot{s}$ is the speed, $\boldsymbol{e}_t$ is the unit tangent vector to the trajectory, and s is the path coordinate along the trajectory.
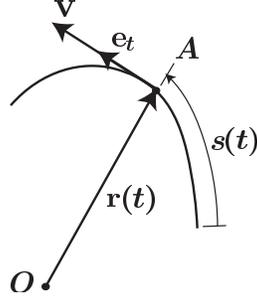
The unit tangent vector can be written as,

$$\boldsymbol{e}_t = \frac{d\boldsymbol{r}}{ds} \ . \tag{3}$$

The acceleration vector is the derivative of the velocity vector with respect to time. Since $\frac{d\boldsymbol{r}}{ds}$ depends only on s, using the chain rule we can write

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \ddot{s}\frac{d\boldsymbol{r}}{ds} + \dot{s}\frac{d}{dt}\left(\frac{d\boldsymbol{r}}{ds}\right) = \ddot{s}\frac{d\boldsymbol{r}}{ds} + \dot{s}^2\frac{d^2\boldsymbol{r}}{ds^2} = \dot{v}\frac{d\boldsymbol{r}}{ds} + v^2\frac{d^2\boldsymbol{r}}{ds^2} \ . \tag{4}$$

The second derivative $d^2\boldsymbol{r}/ds^2$ is another property of the path. We shall see that it is relate to the radius of curvature.

Taking the time derivative of Equation (2), an alternate expression can be written in terms of the unit vector $\boldsymbol{e}_t$ as

$$\boldsymbol{a} = \frac{dv}{dt}\boldsymbol{e}_t + v\frac{d\boldsymbol{e}_t}{dt} \ . \tag{5}$$

The vector $\boldsymbol{e}_t$ is the local unit tangent vector to the curve which changes from point to point. Consequently, the time derivative of $\boldsymbol{e}_t$ will, in general, be nonzero.

The time derivative of $\boldsymbol{e}_t$ can be written as,

$$\frac{d\boldsymbol{e}_t}{dt} = \frac{d\boldsymbol{e}_t}{ds}\frac{ds}{dt} = \frac{d\boldsymbol{e}_t}{ds}v \ . \tag{6}$$

In order to calculate the derivative of $\boldsymbol{e}_t$, we note that, since the magnitude of $\boldsymbol{e}_t$ is constant and equal to one, the only changes that $\boldsymbol{e}_t$ can have are due to rotation, or *swinging*.
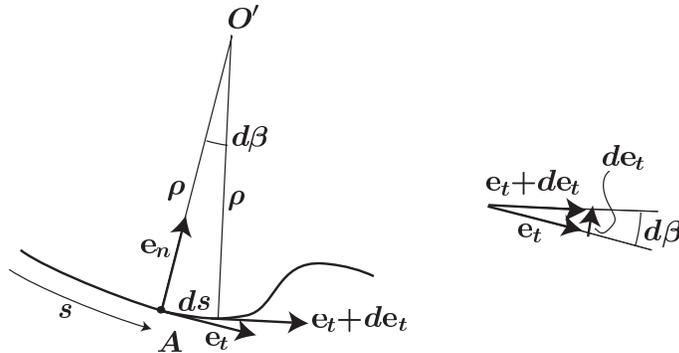
When we move from s to $s + ds$, the tangent vector changes from $\boldsymbol{e}_t$ to $\boldsymbol{e}_t + d\boldsymbol{e}_t$. The change in direction can be related to the angle $d\beta$.

The direction of $d\boldsymbol{e}_t$, which is perpendicular to $\boldsymbol{e}_t$, is called the *normal* direction. On the other hand, the magnitude of $d\boldsymbol{e}_t$ will be equal to the length of $\boldsymbol{e}_t$ (which is one), times $d\beta$. Thus, if $\boldsymbol{e}_n$ is a *unit normal vector* in the direction of $d\boldsymbol{e}_t$, we can write

$$d\boldsymbol{e}_t = d\beta\, \boldsymbol{e}_n \ . \tag{7}$$

Dividing by ds yields,

$$\frac{d\boldsymbol{e}_t}{ds} = \frac{d\beta}{ds}\boldsymbol{e}_n = \kappa \boldsymbol{e}_n = \frac{1}{\rho}\boldsymbol{e}_n \ . \tag{8}$$

Here, $\kappa = d\beta/ds$ is a local property of the curve, called the *curvature*, and $\rho = 1/\kappa$ is called the *radius of curvature*.

Note that in the picture, the sizes of $d\boldsymbol{e}_t$, ds, and $d\beta$ are exaggerated for illustration purposes and actually represent the changes in the limit as ds (and also dt) approach zero.

Note   ***Curvature and radius of curvature***

We consider here two tangent vectors $\boldsymbol{e}_t$ and $\boldsymbol{e} + d\boldsymbol{e}_t$, separated by a small ds and having an angle between them of $d\beta$. If we draw perpendiculars to these two vectors, they will intersect at a point, say, O'. Because the two lines meeting at O' are perpendicular to each of the tangent vectors, the angle between them will be the same as the angle between $\boldsymbol{e}_t$ and $\boldsymbol{e} + d\boldsymbol{e}_t$, $d\beta$. The point O' is called the *center of curvature*, and the distance, ρ, between O' and A is the radius of curvature. Thus, from the sketch, we have that $ds = \rho\, d\beta$, or $d\beta/ds = \kappa = 1/\rho$.
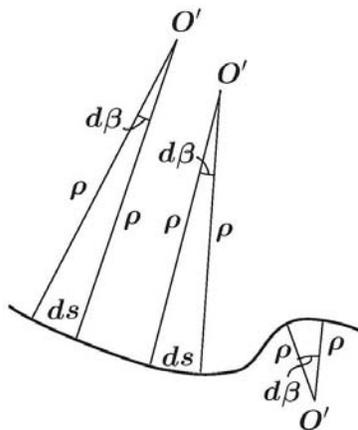
Intuitively, we can see that each infinitesimal arc, ds, can be represented by a circle segment of radius ρ having its center at the center of curvature. It is clear that both the radius of curvature and the center of curvature are functions of s, and consequently they change from point to point.

There are two limiting cases which are of interest. When the trajectory is a circle, the center of curvature does not change and coincides with the center of the circle, and the radius of curvature is equal to the radius

of the circle. On the other hand, when the trajectory is a straight line, the curvature is zero, and the radius of curvature is infinite. Note also, that, in this case, the derivative of e_t is always zero, and the normal direction is not defined.

Going back to expression (6), we have that

$$\frac{de_t}{dt} = \frac{d\beta}{ds} v\, e_n = \dot{\beta}\, e_n = \frac{v}{\rho}\, e_n \ . \tag{9}$$

Finally, we have that the acceleration, can be written as

$$a = \frac{dv}{dt} e_t + \frac{v^2}{\rho} e_n = a_t e_t + a_n e_n \ . \tag{10}$$
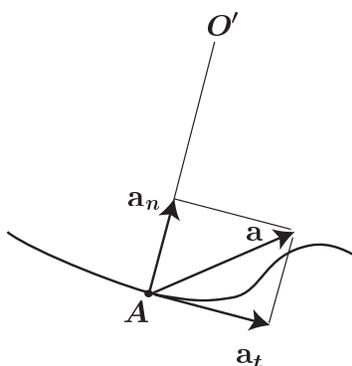
Here, $a_t = \dot{v}$, is the *tangential* component of the acceleration, and $a_n = v^2/\rho$, is the *normal* component of the acceleration. Since a_n is the component of the acceleration pointing towards the center of curvature, it is sometimes referred to as *centripetal* acceleration. When a_t is nonzero, the velocity vector changes magnitude, or *stretches*. When a_n is nonzero, the velocity vector changes direction, or *swings*. The modulus of the total acceleration can be calculated as $a = \sqrt{a_t^2 + a_n^2}$.

Example

A ball is ejected horizontally from the tube with a speed v_0. The only acceleration on the ball is due to gravity. We want to determine the radius of curvature of the trajectory just after the ball is released.
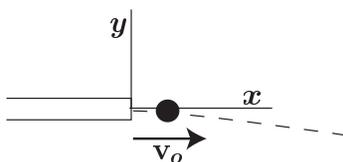
The simplest way to determine the radius of curvature is to note that, initially, the only nonzero component of the acceleration will be in the normal direction, i.e. $a_n = g$. Thus, from $a_n = v_0^2/\rho$, we have that,

$$\rho = \frac{v_0^2}{g} \ .$$

Alternatively, we can obtain an equation for the trajectory of the form $y = f(x)$ and use expression (11) to calculate the curvature. The trajectory is given as,

$$
\begin{aligned}
x &= v_0 t \\
y &= -\frac{1}{2} g t^2 \ .
\end{aligned}
$$

Thus, eliminating t, we have

$$
y = -\frac{g}{2 v_0^2} x^2 \ .
$$

At $x = 0$, $dy/dx = 0$, $d^2 y/dx^2 = -g/v_0^2$, and the above expression gives, $\rho = v_0^2/g$, as expected.


Note   ***Relationship between s, v and a_t***

The quantities s, v and a_t are related in the same manner as the quantities s, v and a for rectilinear motion. In particular we have that $v = \dot{s}$, $a_t = \dot{v}$, and $a_t\, ds = v\, dv$. This means that if we have a way of knowing a_t, we may be able to integrate the tangential component of the motion independently. We will be exploiting these relations in the future.

The vectors $\boldsymbol{e}_t$ and $\boldsymbol{e}_n$, and their respective coordinates t and n, define two orthogonal directions. The plane defined by these two directions, is called the *osculating* plane. This plane changes from point to point, and can be thought of as the plane that locally contains the trajectory (Note that the tangent is the current direction of the velocity, and the normal is the direction into which the velocity is changing).

In order to define a right-handed set of axes we need to introduce an additional unit vector which is orthogonal to $\boldsymbol{e}_t$ and $\boldsymbol{e}_n$. This vector is called the *binormal*, and is defined as $\boldsymbol{e}_b = \boldsymbol{e}_t \times \boldsymbol{e}_n$.
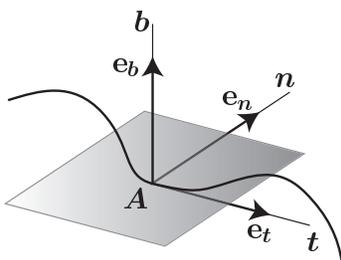
At any point in the trajectory, the position vector, the velocity and acceleration can be referred to these axes. In particular, the velocity and acceleration take very simple forms,

$$
\begin{aligned}
\boldsymbol{v} &= v \boldsymbol{e}_t \\
\boldsymbol{a} &= \dot{v} \boldsymbol{e}_t + \frac{v^2}{\rho} \boldsymbol{e}_n \ .
\end{aligned}
$$

The difficulty of working with this reference frame stems from the fact that the orientation of the axis depends on the trajectory itself. The position vector, $\boldsymbol{r}$, needs to be found by integrating the relation $d\boldsymbol{r}/dt = \boldsymbol{v}$ as

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follows,

$$\boldsymbol{r} = \boldsymbol{r}_0 + \int_0^t \boldsymbol{v} \, dt \; ,$$

where $\boldsymbol{r}_0 = \boldsymbol{r}(0)$ is given by the initial condition.

We note that, by construction, the component of the acceleration along the binormal is always zero.

When the trajectory is planar, the binormal stays constant (orthogonal to the plane of motion). However, when the trajectory is a space curve, the binormal changes with s. It can be shown (see note below) that the derivative of the binormal is always along the direction of the normal. The rate of change of the binormal with s is called the *torsion*, τ. Thus,

$$\frac{d\boldsymbol{e}_b}{ds} = -\tau \, \boldsymbol{e}_n \qquad \text{or,} \qquad \frac{d\boldsymbol{e}_b}{dt} = -\tau v \, \boldsymbol{e}_n.$$

We see that whenever the torsion is zero, the trajectory is planar, and whenever the curvature is zero, the trajectory is linear.

Note   ***Calculation of the radius of curvature and torsion for a trajectory***

In some situations the trajectory will be known as a curve of the form $y = f(x)$. The radius of curvature in this case can be computed according to the expression,

$$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{|d^2y/dx^2|} \; . \tag{11}$$

This expression is not hard to derive. Try it! Since $y = f(x)$ defines a planar curve, the torsion τ is zero. On the other hand, if the trajectory is known in parametric form as a curve of the form $\boldsymbol{r}(t)$, where t can be time, but also any other parameter, then the radius of curvature ρ and the torsion τ can be computed as

$$\rho = \frac{(\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}})^{3/2}}{\sqrt{(\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}})(\ddot{\boldsymbol{r}} \cdot \ddot{\boldsymbol{r}}) - (\dot{\boldsymbol{r}} \cdot \ddot{\boldsymbol{r}})^2}} \; ,$$

where $\dot{\boldsymbol{r}} = d\boldsymbol{r}/dt$, and $\ddot{\boldsymbol{r}} = d^2\boldsymbol{r}/dt^2$ and

$$\tau = \frac{(\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}}) \cdot \dddot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}}|^2} \tag{12}$$

where $\dddot{\boldsymbol{r}} = d^3r/dt^3$.

Equations of Motion in Intrinsic Coordinates

Newton's second law is a vector equation, $\boldsymbol{F} = m\boldsymbol{a}$, which can now be written in intrinsic coordinates. In tangent, normal and binormal components, tnb, we write $\boldsymbol{F} = F_t\boldsymbol{e}_t + F_n\boldsymbol{e}_n$ and $\boldsymbol{a} = a_t\boldsymbol{e}_t + a_n\boldsymbol{e}_n$. We observe that the positive direction of the normal coordinate is that pointing towards the center of curvature. Thus, in component form, we have

$$\begin{aligned} F_t &= m\,a_t = m\,\dot{v} = m\,\ddot{s} \\ F_n &= m\,a_n = m\,\frac{v^2}{\rho} \end{aligned}$$

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Note that, by definition, the component of the acceleration along the binormal direction, e_b, is always zero, and consequently the binormal component of the force must also be zero. This may seem surprising, at first, but recall that the tangent and normal directions are determined by the motion, and, hence, we can say that the motion "chooses" the binormal direction to be always orthogonal to the applied force. In other words, if we apply a force to a particle, the particle will experience an acceleration which is parallel to the force. The normal direction is chosen so that the acceleration vector is always contained in the plane defined by the tangent and the normal. Thus, the binormal is always orthogonal to the external force.

Intrinsic coordinates are sometimes useful when we are dealing with problems in which the motion is constrained, such as a car on a roller coaster. The geometry of the trajectory is known, and, therefore, the directions of the tangent, normal and binormal vectors are also known. In these cases it may be possible to integrate the component of the equation of motion along the tangential direction (especially if there is no friction), and then calculate, a-posteriori, the reaction force using the normal component of the equation of motion.

Note (optional)   ***Frenet formulae***

The Frenet formulae give us the variations of the unit vectors e_t, e_n and e_b with respect to the path coordinate s. The first formula

$$\frac{de_t}{ds} = \frac{1}{\rho}e_n \ ,$$

has already been defined. Now, since e_b is a unit vector, de_b/ds will be orthogonal to e_b. Hence, it will be of the form,

$$\frac{de_b}{ds} = b_t e_t + b_n e_n \ .$$

If we perform the dot product of this expression with e_t, we obtain

$$b_t = \frac{de_b}{ds} \cdot e_t = -\frac{de_t}{ds} \cdot e_b = -\frac{1}{\rho}e_n \cdot e_b = 0 \ .$$

The second equality follows from the fact that the derivative of $e_t \cdot e_b$ is zero, i.e. $de_t \cdot e_b + e_t \cdot de_b = 0$. Therefore, only b_n is nonzero. Defining $b_n = -\tau$, we obtain the third Frenet formula

$$\frac{de_b}{ds} = -\tau e_n \ .$$

Finally, the second formula can be obtained in a similar manner (we leave the details as an exercise) and gives,

$$\frac{de_n}{ds} = -\frac{1}{\rho}e_t + \tau e_b \ ,$$

or, multiplying by v,

$$\frac{de_n}{dt} = -\frac{v}{\rho}e_t + \tau v e_b \ . \tag{13}$$

As we move along s, the osculating plane (and hence e_b) may rotate, making the curve non planar. As an example, an aeroplane may be rolling as it flies along e_t. The derivative of e_b is in the direction opposite to

e_n if the rotation is in the direction of a right hand screw, and this is taken as the positive direction for the torsion.


Example   *"Simplified" Aircraft Kinematics (W. M. Hollister)*

The flight of an aircraft through the sky is an example of curvilinear motion. Think of e_t as the roll axis aligned with the velocity vector of the aircraft. Think of e_b as being the pitch axis. The lift is then directed along e_n. The roll rate of the aircraft can be interpreted as τv, and the pitch rate as v/ρ. In order to turn the aircraft out of the vertical plane, it is necessary to rotate the direction of the lift e_n so that there is a component of acceleration out of the vertical plane. Neglecting gravity, the velocity vector along e_t determines where the aircraft is going, and the lift along e_n determines where the velocity vector is going. The roll rate determines how the lift vector will be rotated out of the osculating plane. As shown by equation (13), the direction of the lift vector is changed by rolling τv, as well as pitching v/ρ.

Consider the following example. An aircraft follows a spiral path in the sky while doing a barrel roll. The coordinates are given below, where $v_0 = 194$ ft/s, $\omega = 0.4$ rad/s, and $h = 125$ ft are constants.

$$
\begin{aligned}
x &= v_0 t \\
y &= h \cos \omega t \\
z &= h \sin \omega t
\end{aligned}
$$
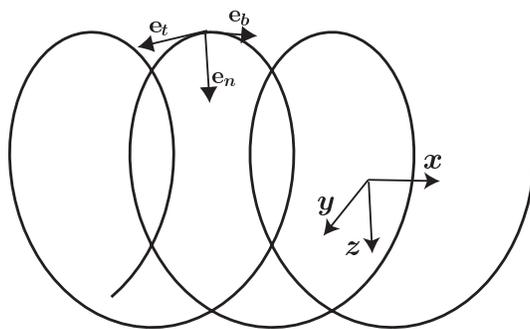
We have,

$$
\begin{aligned}
\boldsymbol{v} &= v_0 \boldsymbol{i} - h\omega \sin \omega t \boldsymbol{j} + h\omega \cos \omega t \boldsymbol{k} \\
v &= \sqrt{v_0^2 + h^2\omega^2} = 200 \text{ft/s} \approx v_0 \\
\boldsymbol{a} &= -h\omega^2 \cos \omega t \boldsymbol{j} - h\omega^2 \sin \omega t \boldsymbol{k} \\
a &= h\omega^2 = 20 \text{ft/s}^2
\end{aligned}
$$

Since $\dot{v} = 0$, $\boldsymbol{a} = (v^2/\rho)\boldsymbol{e}_n$, or,

$$
h\omega^2 = \frac{v^2}{\rho}, \qquad \rho = \frac{v^2}{h\omega^2} = 2000 \text{ft}
$$

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,

$$\begin{array}{rcl}
\boldsymbol{e}_t &=& \dfrac{v_0}{v}\boldsymbol{i} - \dfrac{h\omega}{v}\sin\omega t\boldsymbol{j} + \dfrac{h\omega}{v}\cos\omega t\boldsymbol{k} \\[2mm]
\boldsymbol{e}_n &=& -\cos\omega t\boldsymbol{j} - \sin\omega t\boldsymbol{k} \\[2mm]
\boldsymbol{e}_b &=& \dfrac{h\omega}{v}\boldsymbol{i} + \dfrac{v_0}{v}\sin\omega t\boldsymbol{j} - \dfrac{v_0}{v}\cos\omega t\boldsymbol{k}
\end{array}$$

Finally,

$$\frac{d\boldsymbol{e}_b}{dt} = \frac{v_0\omega}{v}\cos\omega t\boldsymbol{j} + \frac{v_0\omega}{v}\sin\omega t\boldsymbol{k}$$

which corresponds to a roll rate of

$$\tau v = \frac{v_0\omega}{v} \approx \omega = 0.4\mathrm{rad/s}$$

Example   **Aircraft flying an "imperfect" loop** (Hollister)

It is more common to fly a loop keeping the normal acceleration, a_n, approximately constant at, say, ng ($n \sim 3 - 4$).

Let β be the flight path angle which the velocity vector (or tangent) makes with the horizontal, and let ρ be the radius of curvature of the path. Then,

$$\begin{array}{rcl}
a_n &=& \dfrac{v^2}{\rho} = v\dot\beta = ng \\[2mm]
a_t &=& -g\sin\beta \ .
\end{array}$$

From, $v\,dv = a_t\,ds$, with $ds = \rho d\beta$, we have,

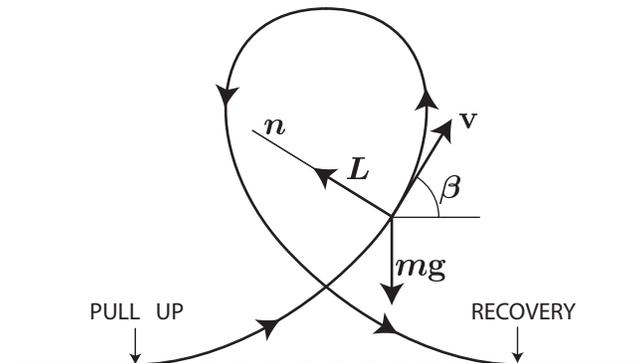$$v\,dv = a_t\rho\,d\beta = (-g\sin\beta)(\frac{v^2}{ng})d\beta \ ,$$

or, integrating,

$$\frac{n\,dv}{v} = -\sin\beta d\beta \ , \qquad n\ln v|_{v_0}^{v} = \cos\beta|_0^\beta \ , \qquad v = v_0 e^{-\frac{1}{n}(1-\cos\beta)} \ ,$$

and

$$\rho = \frac{v_0^2}{ng}e^{-\frac{2}{n}(1-\cos\beta)} \ . \tag{14}$$

A sketch of this path is shown in the figure below.

In going over the top of the imperfect loop, the aircraft does not go as high or loose as much velocity as it does going over the perfect loop. Unlike the perfect loop case however, the aircraft does need to pull up before, and recover after, the point of maximum altitude (see diagram). Note that the solution of this example and the previous one would have been rather difficult using rectangular coordinates. Note also that the form of the solution given by equation (14) is rather unusual, e.g. the radius of curvature is given as a function of the attitude angle. A possible way to plot the trajectory starting from an initial position and velocity would be to first determine β, and then draw a small circle segment with the appropriate ρ. After calculating the new position, the process would be repeated to draw the entire trajectory.

Example   ***Pursuit Problem***

A classic problem in particle kinematics is the pursuit problem. This has an elegant history in mathematics and practical applications in air combat. At its most basic, we have a "target" **A** traveling on a known curve at constant speed. The target is pursued by **B** whose strategy is to turn such that her velocity vector is always pointed at the target.
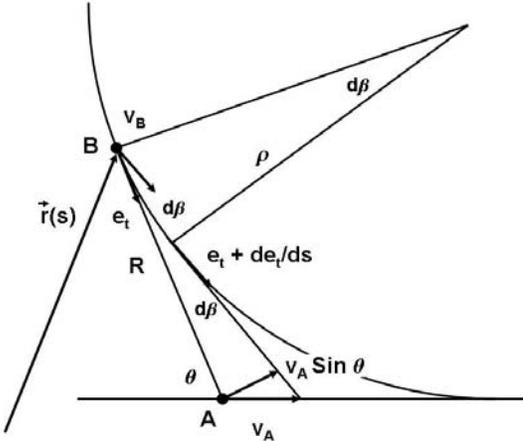
. We take the simplest version of this problem where the velocities of the two vehicles are equal: $v_A = v_B$ and the evading vehicle travels on a straight line in the positive x direction. Analyzing this problem using intrinsic coordinates can give us some insight into the motion, into the formalism of tangential and normal unit vectors and the role of radius of curvature of the path. We examine this problem at the instant where the distance between **A** and **B** is given by R and the angle between their directions of travel is θ. (i.e. if the particles are moving in the same direction, $\theta = 0$.) We allow a time dt to elapse and examine the resulting geometry. Of course to allow us to see the motion, we take dt to be rather large. Obviously we are talking about a limiting process as $dt -> 0$.

From the figure, we see that in a time dt, the vector **R** between the two points A and B changes direction by an angle $d\beta = v_A \sin\theta / R dt$. Therefore the direction of the tangential vector also changes by $d\beta$. From

our earlier analysis the change of the tangential vector due to a change in ds is given by

$$\frac{d\boldsymbol{e_t}}{ds} = \boldsymbol{e_n}\frac{d\beta}{ds} = \frac{1}{\rho}\boldsymbol{e_n} \tag{15}$$

Therefore, using the chain rule, the magnitude of radius of curvature at the point $\boldsymbol{r}(t)$ is given by

$$\frac{1}{\rho} = \frac{d\beta}{ds} = \frac{\frac{d\beta}{dt}}{\frac{ds}{dt}} \tag{16}$$

Now, ds/dt is the scalar velocity of the point B along the pursuit curve, while $d\beta/dt$ is the rate at which the angle β changes as a result of the motion of point A.

$$\frac{ds}{dt} = v_B \tag{17}$$

$$\frac{d\beta}{dt} = \frac{v_A \sin\theta}{R} \tag{18}$$

so that the radius of curvature at the point A is given by

$$1/\rho = \frac{v_A \sin\theta}{v_B R} \tag{19}$$

Given this, the acceleration of particle B is easily found as $a = v_B^2/\rho$.

ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition

2/5, 3/5 (normal and tangential coordinates only)

MIT OpenCourseWare
http://ocw.mit.edu

16.07 Dynamics
Fall 2009