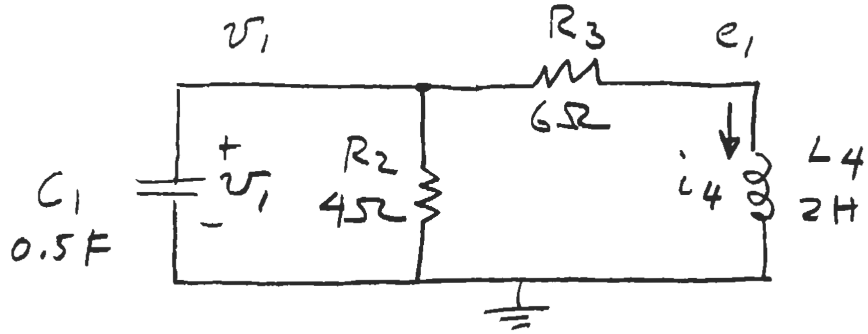
Lecture S12

From last time:



__Step 1__ Identify the states. These are

$v_1(t)$, $i_4(t)$

__Step 2__ Use constitutive laws to write differential equations for $v_1(t)$, $i_4(t)$:

$$i_1 = C_1 \frac{dv_1}{dt} \Rightarrow \frac{dv_1}{dt} = \frac{1}{C_1} i_1$$

$$v_4 = L_4 \frac{di_4}{dt} \Rightarrow \frac{di_4}{dt} = \frac{1}{L_4} v_4$$

__Step 3__ Solve for i_1, v_4 using, say, node method. Treat v_1 and i_4 as __known__:

$$\frac{e_1 - v_1}{R_3} + i_4 = 0 \qquad \text{(KCL at } e_1\text{)}$$

$$\Rightarrow \quad e_1 = v_1 - R_3 i_4$$

But $v_4 = e_1 - 0$.

$$\Rightarrow \frac{di_4}{dt} = \frac{v_4}{L_4} = \frac{v_1}{L_4} - \frac{R_3}{L_4} i_4$$

$$\boxed{\frac{di_4}{dt} = \frac{1}{2} v_1 - 3 i_4}$$

This is in required form:

$$\frac{di_4}{dt} = \dot{x}_2 = f_2(x_1, x_2)$$

Now find i_1. [concept test here]

Solution: We have "solved" network, in that we know all the node voltages. To find current through C_1, apply KCL at v_1 node, since constitutive law does not give current.

$$v_1: \quad i_1 + \frac{v_1}{R_2} + \frac{v_1 - e_1}{R_3} = 0$$

$$\Rightarrow i_1 = -\left(\frac{1}{R_2} + \frac{1}{R_3}\right) v_1 + \frac{1}{R_3} e_1$$

$$= -\left(\frac{1}{R_2} + \frac{1}{R_3}\right) v_1 + \frac{1}{R_3} (v_1 - R_3 i_4)$$

$$= -\frac{1}{R_2} v_1 - i_4$$

Therefore,

$$\frac{dv_1}{dt} = \frac{1}{C_1} i_1 = -\frac{1}{R_2 C_1} v_1 - \frac{1}{C_1} i_4$$

$$\boxed{\frac{dv_1}{dt} = -\frac{1}{2} v_1 - 2 i_4}$$

Therefore,

$$\dot{\underline{x}} = \frac{d}{dt}\begin{pmatrix} v_1 \\ i_4 \end{pmatrix} = \begin{pmatrix} -1/2 & -2 \\ 1/2 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ i_4 \end{pmatrix}$$

$$= A \underline{x}$$

$$A = \begin{bmatrix} -1/2 & -2 \\ 1/2 & -3 \end{bmatrix}$$

Note that differential equations have the form

$$\dot{\underline{x}} = A\underline{x}$$

$\underline{x}$ = state vector

A = state dynamics matrix

This form of state equation is very common, and very useful. Any linear, homogeneous system has dynamics in this form.

Why use state-space approach?

- Approach is very general
- There is a rich theory describing solutions
- There are good numerical algorithms for obtaining solutions
- Approach is central to "modern" control.

So what is $v_1(t)$, $i_4(t)$, if

$$v_1(0) = 2\ \text{V}, \quad i_4(0) = 1\ \text{A} \quad ?$$

First, find *general* solution to

$$\dot{\underline{x}} = A\underline{x}$$

What is solution? Guess!

$$\underline{x} = \underline{v}\, e^{st}$$

Plug this in:

$$\Rightarrow \quad \underline{v}\, s \cancel{e^{st}} = A\underline{v}\cancel{e^{st}}$$

$$\Rightarrow \quad s\underline{v} = A\underline{v} \qquad \text{or} \qquad (sI - A)\underline{v} = \underline{0}$$

A (nontrivial) solution $\underline{v}$ to this equation is an <u>eigenvector</u>; the corresponding value of s is an <u>eigenvector</u>.

These are special cases of characteristic vectors and characteristic values.

[Do concept test]

To find eigenvalues, set

$$\phi(s) = \det(sI - A) = 0$$

$$\det(sI - A) = \begin{vmatrix} s + 1/2 & +2 \\ -1/2 & s+3 \end{vmatrix}$$

$$= s^2 + 3.5s + 2.5 = (s-1)(s-2.5)$$

So $\lambda_1 = -1$
$\lambda_2 = -2.5$

To find eigenvectors, set $(\lambda I - A)\underline{v} = 0$

$$\lambda_1 I - A = \begin{bmatrix} -1/2 & 2 \\ -1/2 & 2 \end{bmatrix}$$

$$(\lambda_1 I - A)\underline{v}_1 = \underline{0} \Rightarrow \underline{v}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

Similarly, $\underline{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

So general solution is

$$\underline{x}(t) = \underline{v}_1 e^{\lambda_1 t} + \underline{v}_2 e^{\lambda_2 t}$$