LECTURE S6

Useful Properties of Unit Impulse

Have already derived that

$$\delta(t) = \frac{d}{dt}\sigma(t)$$

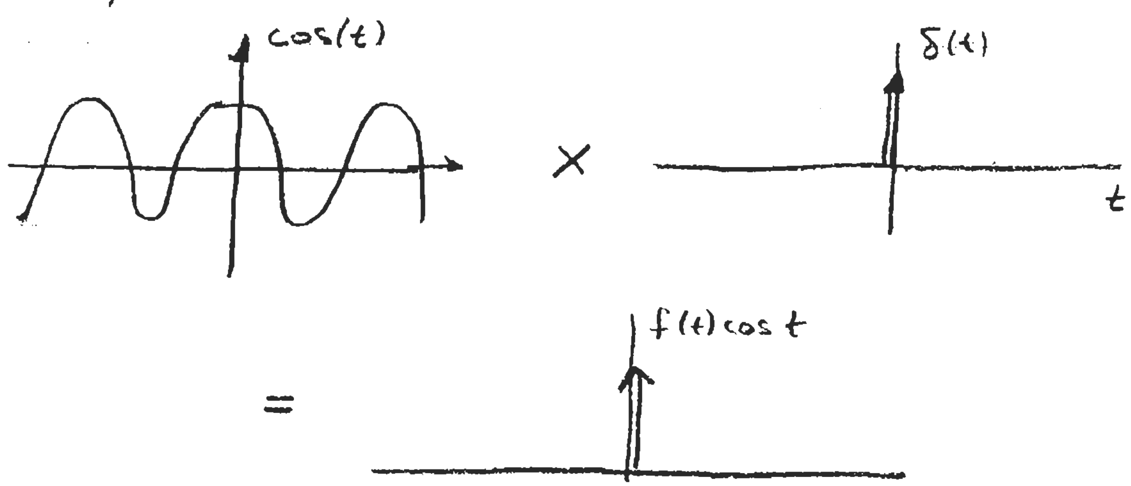
$$\Rightarrow \quad \sigma(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

Recall that $\delta(t) = 0, \quad t \neq 0$. Therefore,

$$f(t)\,\delta(t) = f(0)\,\delta(t)$$

if $f(t)$ "well-behaved" near $t=0$.

Example: $f(t) = \cos t$



$$\delta(t)\cos t = \delta(t)\cos(0) = \delta(t)$$

<u>Example</u>: $\sigma(t)\,\delta(t) = ?$

$= $ undefined

because $\sigma(t)$ is discontinuous at $t=0$

Let's find $f(t) * \delta(t)$, $\delta(t) * f(t)$:

$$f(t) * \delta(t) = \int_{-\infty}^{\infty} f(t-\tau)\,\delta(\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} f(t)\,\delta(\tau)\,d\tau$$

value of $f(t-\tau)$ @ $\tau=0$

$$= f(t)\underbrace{\int_{-\infty}^{\infty} \delta(\tau)\,d\tau}_{\text{area}=1} = f(t)$$

$$\delta(t) * f(t) = \int_{-\infty}^{\infty} \underbrace{\delta(t-\tau)}_{\text{impulse at } \tau=t} f(\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} \delta(t-\tau)\,f(t)\,d\tau$$

value at $\tau = t$

$$= f(t) \int_{-\infty}^{\infty} \delta(t-\tau)\, d\tau = f(t)$$

So,

$$f(t) * \delta(t) = f(t) \qquad \xrightarrow{\delta(t)} \boxed{F} \xrightarrow{f(t)}$$

"The response of a sytem, which has impulse response $f(t)$, to an impulse is $f(t)$"

$$\delta(t) * f(t) = f(t)$$

"The response of a system, which has impulse response $g(t) = \delta(t)$, to an input $f(t)$ is $f(t)$"

Properties of Convolution

Commutative Property:

$$f(t) * g(t) = g(t) * f(t)$$

Proof:

$$f * g = \int_{-\infty}^{\infty} f(t-\tau)\, g(\tau)\, d\tau$$

Let $\tau_2 = t - \tau$. Then

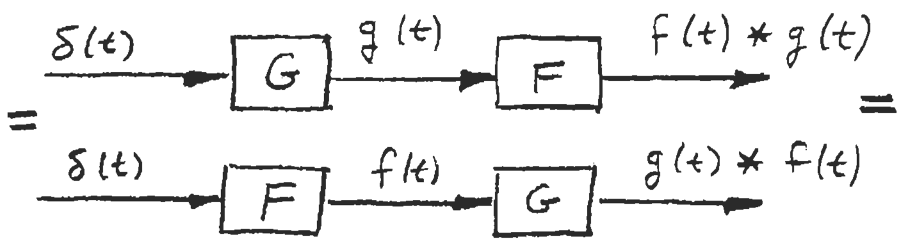
$$d\tau_2 = -d\tau \qquad (t = \text{const. in integral})$$

$$\Rightarrow f * g = -\int_{+\infty}^{-\infty} f(\tau_2)\, g(t-\tau_2)\, d\tau_2$$

$$= \int_{-\infty}^{\infty} g(t-\tau) f(\tau)\, d\tau$$

$$= g * f \qquad \text{Q.E.D.}$$

Block Diagram Interpretation:



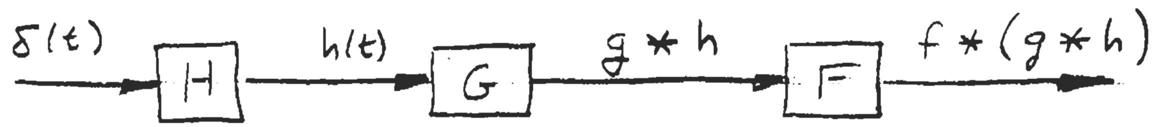
So order of blocks is unimportant.

<u>Associative Property</u>:

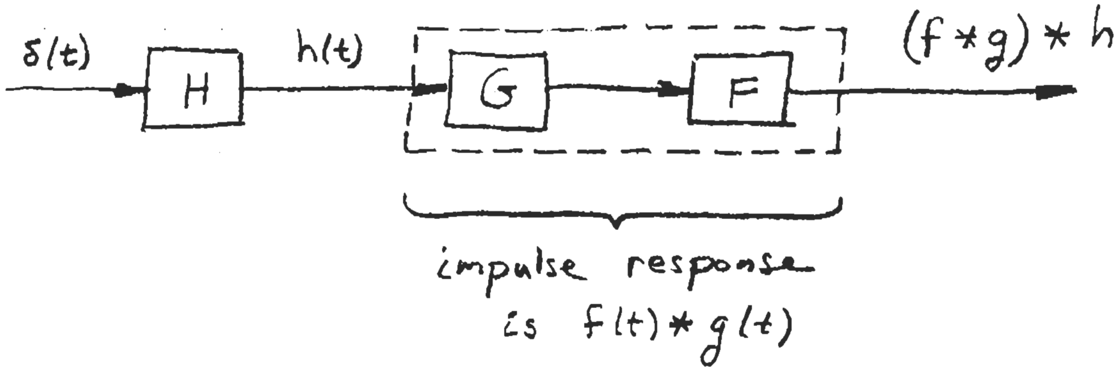
$$(f * g) * h = f * (g * h)$$

<u>Proof</u> Write out two double integrals, change order of integration, show that they are the same (ugh!).

<u>Or</u>, use block diagram argument:



Same system:



Because convolution is associative, we can write $f * g * h$ without any parentheses.