

# M12 Buckling of Simple Columns

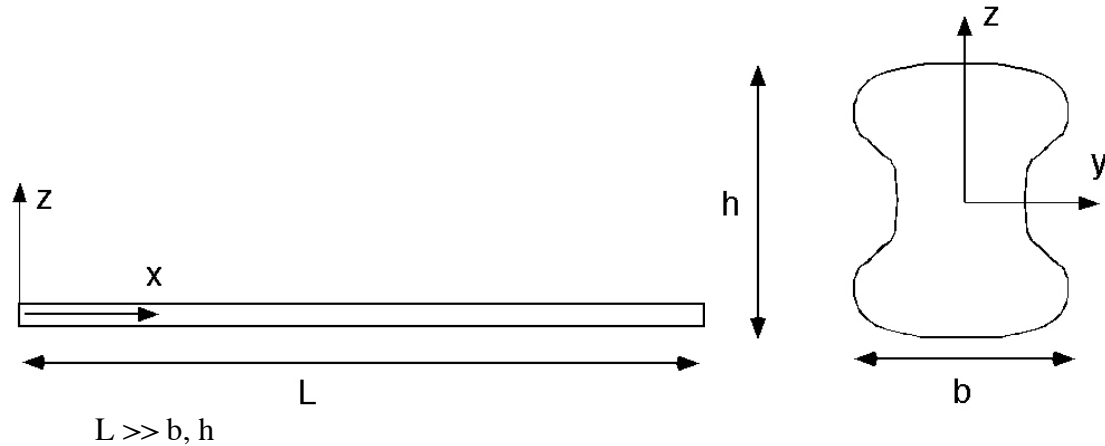
Reading Crandall, Dahl and Lardner: 9.4

Now extend the idea of elastic instabilities to a continuous structural member.

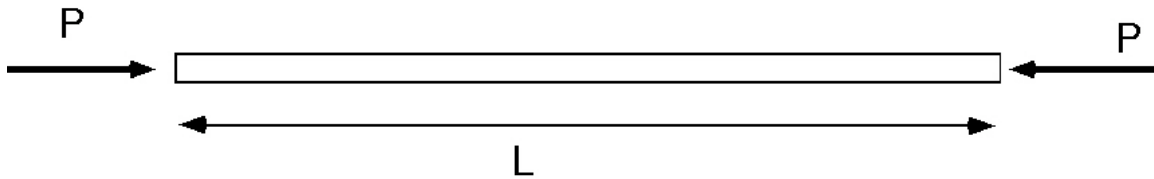
Model of a column

A column is a slender structural member that carries axial compressive loads.

(a) Geometry – identical to a beam, long, straight, slender, symmetric cross-section etc.



(b) Loading – axial compressive axial forces

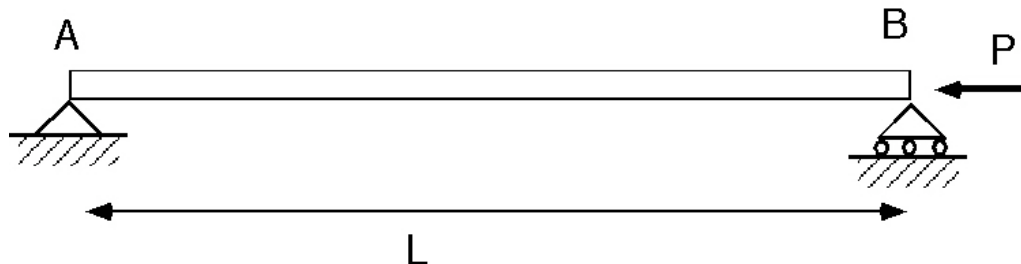


(c.) Deformations

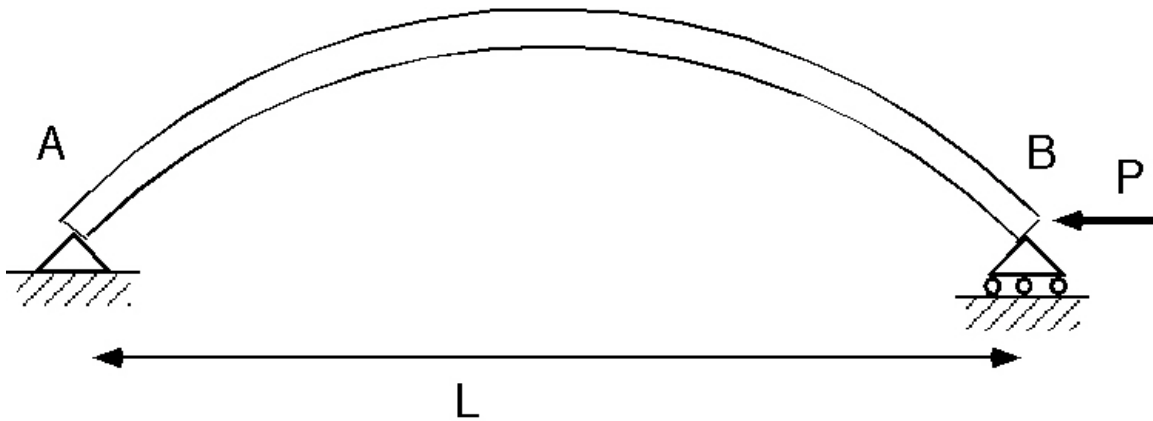
At low loads - same as a rod, axial (x) stresses, axial deformation only

At higher loads - buckling deflection (transverse) governed by bending relations

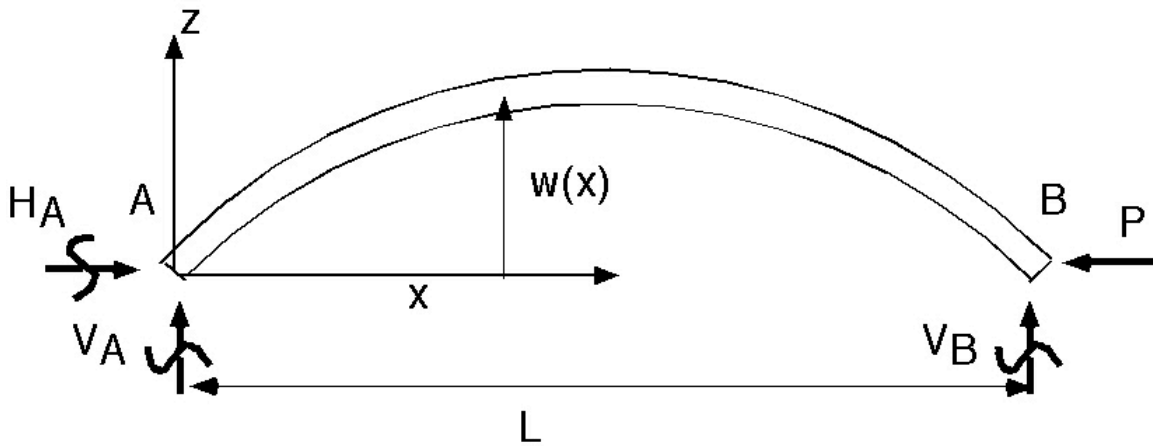
$$\text{i.e., } \frac{d^2 w}{dx^2} = \frac{M}{EI}$$



deflected shape:

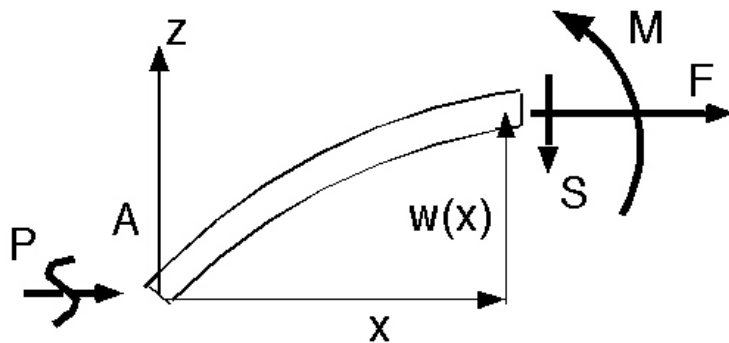


Draw free body diagram of deflected column:



From equilibrium:  $H_A = P$ ,  $V_A = V_B = 0$

We can now use method of sections, by taking a “cut” at an arbitrary x position (as for a beam)



By equilibrium

$$\begin{aligned} \square F_x = 0 \quad \square \quad P + F = 0 \quad \square \quad F = -P \\ \square F_y = 0 \quad \uparrow + \quad S = 0 \\ \square M_A = 0 \quad \curvearrowright + \quad M - Pw(x) = 0 \\ \quad \quad \quad \quad \quad M + Pw(x) = 0 \end{aligned}$$

Substitute for the moment, M, using the moment-curvature relationship, obtain second order differential equation

$$EI \frac{d^2 w}{dx^2} + Pw = 0$$

Stabilizing  
(restoring)

Tends to destabilize -  
compressive load  
as w increases

This is the governing differential equation for Euler buckling (buckling of simple columns)

Solution to Euler buckling

$$w = e^{\lambda x} \quad (\text{recall from 18.03})$$

rewrite as:

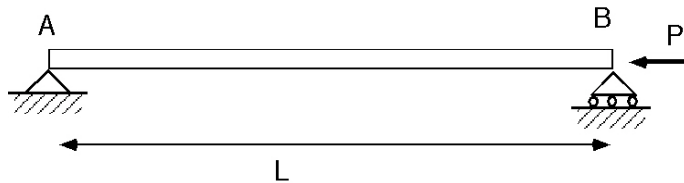
$$\begin{aligned} \frac{d^2 w}{dx^2} + \frac{P}{EI} w = 0 \\ \lambda^2 e^{\lambda x} + \frac{P}{EI} e^{\lambda x} = 0 \\ \lambda^2 = -\frac{P}{EI} \quad \square \quad \lambda = \pm \sqrt{\frac{P}{EI}} i \end{aligned}$$

where  $i = \sqrt{-1}$

The physical significance of this solution will be more apparent if we express as sine and cosine terms. Hence we can express the complete homogeneous solution as:

$$w = A \sin \sqrt{\frac{P}{EI}} x + B \cos \sqrt{\frac{P}{EI}} x + Cx + D$$

For the simply supported case:



at  $x=0$ ,  $M=0$  (pinned supports)

$$M = EI \frac{d^2 w}{dx^2} = 0 \quad \frac{d^2 w}{dx^2} = A \frac{P}{EI} \sin \sqrt{\frac{P}{EI}} x - B \frac{P}{EI} \cos \sqrt{\frac{P}{EI}} x$$

$$\frac{d^2 w}{dx^2}(x=0) = 0 \quad B = 0$$

and  $w=0$  at  $x=0$

$$w = 0 = +B + D$$

$$\therefore D = 0$$

Similarly at  $x=L$

$$w(x=L) = 0 \quad +A \sin \sqrt{\frac{P}{EI}} L + CL = 0$$

$$\frac{d^2 w}{dx^2}(x=L) = 0 \quad A \frac{P}{EI} \sin \sqrt{\frac{P}{EI}} L = 0$$

So:  $A \sin \sqrt{\frac{P}{EI}} L = 0 \implies C = 0$

This is satisfied if  $A=0$ , but this is not very interesting.

which leaves:  $\sin \sqrt{\frac{P}{EI}} L = 0 \implies \sqrt{\frac{P}{EI}} L = n\pi$

integer

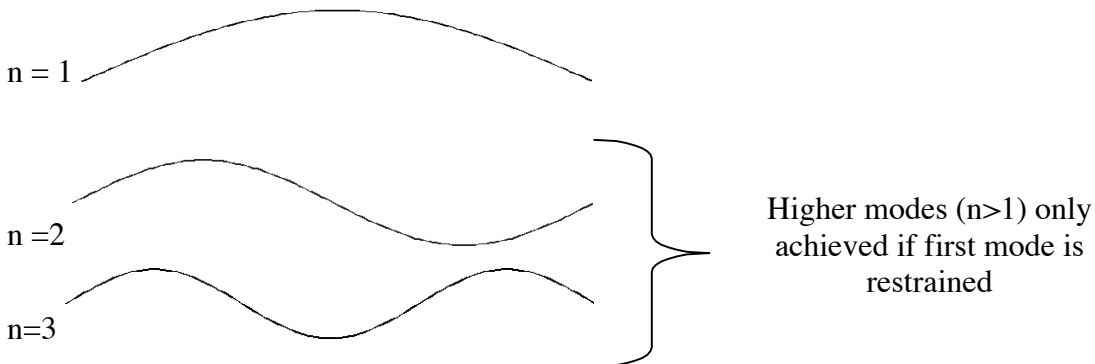
Thus buckling occurs if:

$$P = \frac{n^2 \pi^2 EI}{L^2}$$

and the deformed shape

$$w = A \sin \frac{n\pi x}{L}$$

Note: A is not uniquely defined



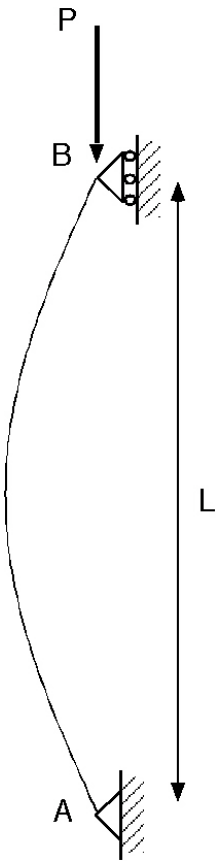
the lowest value is the critical buckling load

$$P_{crit} = \frac{\pi^2 EI}{L^2}$$

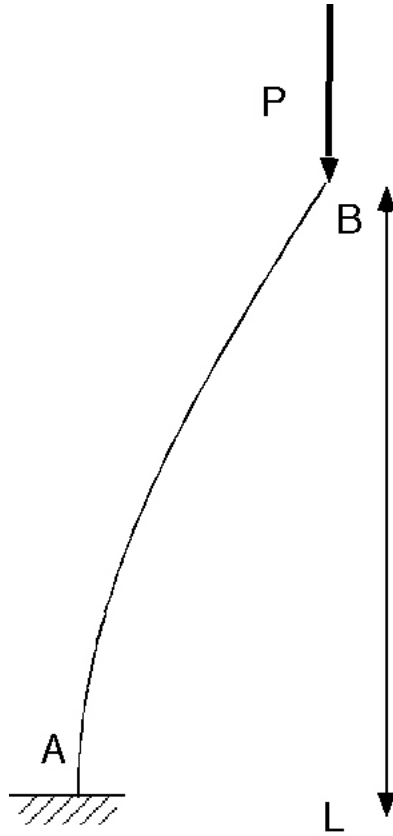
Other boundary conditions provide other values for A, B, C, & D, but the solution is always of the form:

$$w = A \sin \sqrt{\frac{P}{EI}} x + B \cos \sqrt{\frac{P}{EI}} x + Cx + D$$

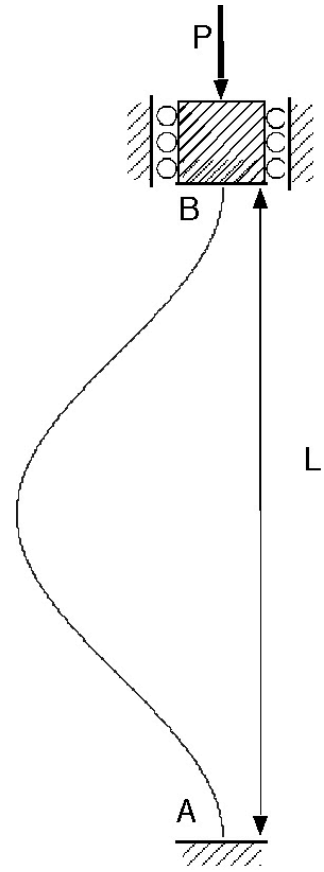
and the buckling load has the form:  $P_{cr} = \frac{c \pi^2 EI}{L^2}$



Simply supported  $c=1$



Clamped-free  $c=1/4$

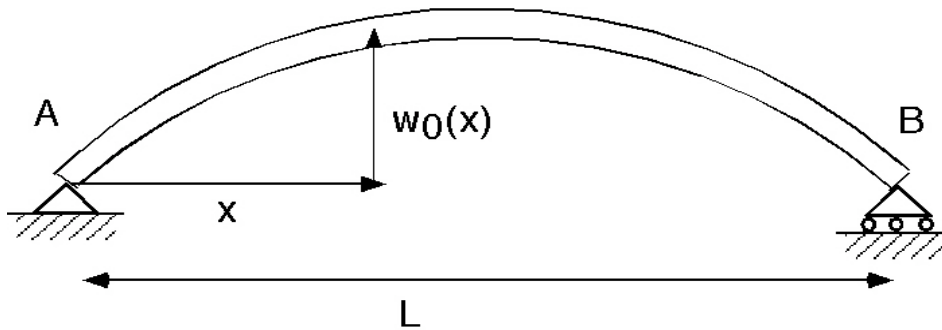


clamped-clamped  $c=4$

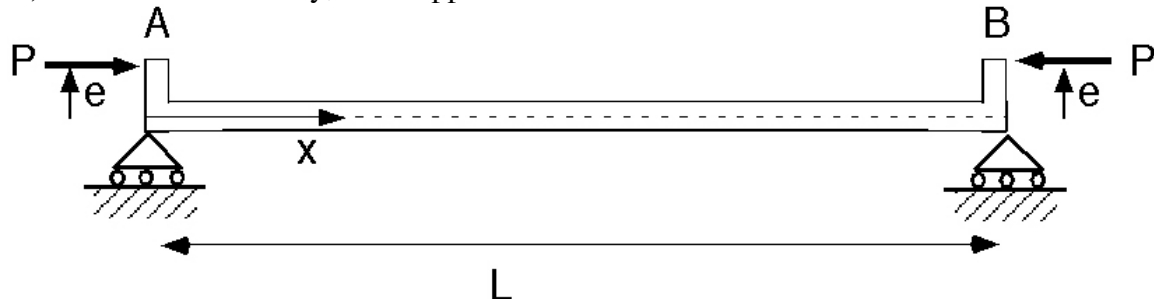
## Effect of Initial Imperfections on Buckling

Up to now we have considered the column to be initially straight and loaded along its axis. In reality a structure and its loading will never match these idealizations. Small deviations from ideal do not matter when we are considering the behavior of beams, shafts and rods in tensions, however they are quite important in determining elastic instabilities. There are two types of imperfection that commonly occur.

- 1.) Shape - initial deflection,  $w_0(x)$  (exaggerated)



- 2.) Load eccentricity, loads applied a distance  $e$  from the neutral axis



Cases (1) and (2) are both handled in the same way, set up governing differential equation and solve with solution of form:

$$w = A \sin \sqrt{\frac{P}{EI}} x + B \cos \sqrt{\frac{P}{EI}} x + Cx + D$$

But these initial imperfections introduce new boundary conditions

Consider case (2)

e.g., at  $x = 0, L$        $M = Pe$

$$x = 0 \quad w = 0$$

$$w = B + D = 0$$

$$M = EI \frac{d^2 w}{dx^2} = Pe = -EI \frac{P}{EI} B$$

Hence  $B = -e$ ,  $D = +e$

$$x = L \quad w = 0$$

$$M = EI \frac{d^2 w}{dx^2} = Pe$$

Solving for A:

$$M = EI \frac{d^2 w}{dx^2} = Pe = -AEI \frac{P}{EI} \sin \sqrt{\frac{P}{EI}} L + EI \frac{P}{EI} e \cos \sqrt{\frac{P}{EI}} L$$

Hence

$$A = \frac{e \cos \sqrt{\frac{P}{EI}} L - 1}{\sin \sqrt{\frac{P}{EI}} L} \quad \text{Note: A is now defined}$$

Solving for C:

$$w(L) = 0 = \frac{e \cos \sqrt{\frac{P}{EI}} L - 1}{\sin \sqrt{\frac{P}{EI}} L} \sin \sqrt{\frac{P}{EI}} L - e \cos \sqrt{\frac{P}{EI}} L + CL + e$$

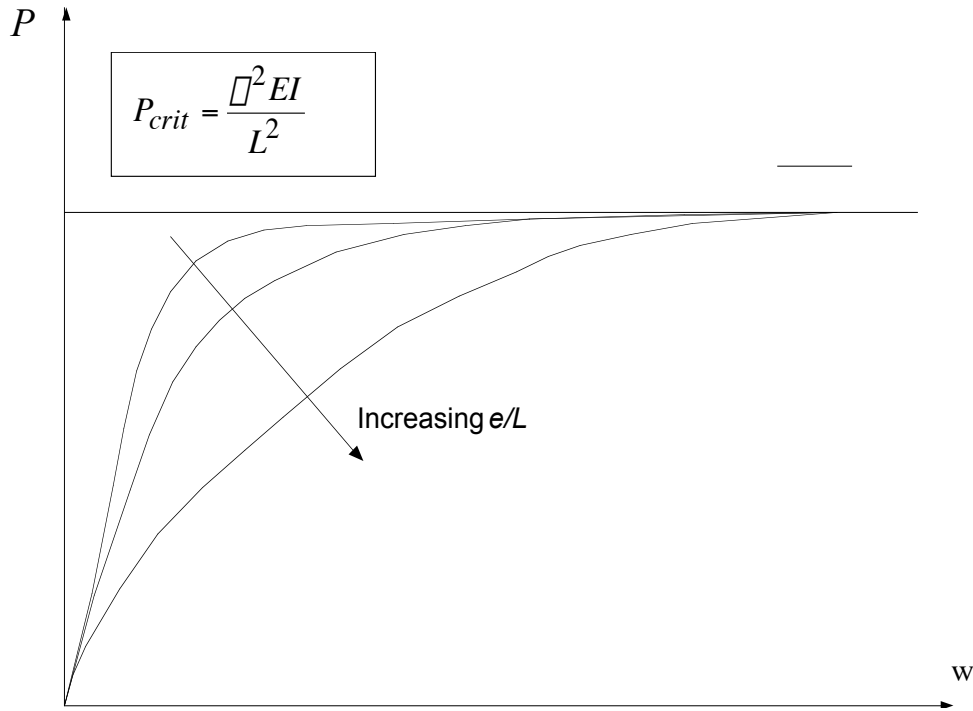
$$w(L) = 0 = CL \quad \text{hence } C = 0$$

combining

$$w = e \frac{\cos \sqrt{\frac{P}{EI}} L - 1}{\sin \sqrt{\frac{P}{EI}} L} \sin \sqrt{\frac{P}{EI}} x + \cos \sqrt{\frac{P}{EI}} x - 1$$

now obtain finite values of  $w$  for all values of  $P$





This behavior is similar to the case we considered in M10/M11 – of the axially loaded rigid rod with the torsional spring at one end.