Fluids – Lecture 4 Notes

1. Dimensional Analysis – Buckingham Pi Theorem
2. Dynamic Similarity – Mach and Reynolds Numbers

Reading: Anderson 1.7

Dimensional Analysis

Physical parameters

A large number of physical parameters determine aerodynamic forces and moments. Specifically, the following parameters are involved in the production of lift.

Parameter	Symbol	Units
Lift per span	L'	mt^{-2}
Angle of attack	α	—
Freestream velocity	V_∞	lt^{-1}
Freestream density	ρ_∞	ml^{-3}
Freestream viscosity	μ_∞	$ml^{-1}t^{-1}$
Freestream speed of sound	a_∞	lt^{-1}
Size of body (e.g. chord)	c	l

For an airfoil of a given shape, the lift per span in general will be a function of the remaining parameters in the above list.

$$L' \;=\; f(\alpha, \rho_\infty, V_\infty, c, \mu_\infty, a_\infty) \tag{1}$$

In this particular example, the functional statement has $N = 7$ parameters, expressed in a total of $K = 3$ units (mass m, length l, and time t).

Dimensionless Forms

The Buckingham Pi Theorem states that this functional statement can be rescaled into an equivalent *dimensionless* statement

$$\Pi_1 \;=\; \bar{f}(\,\Pi_2, \Pi_3 \ldots \Pi_{N-K})$$

having only $N-K = 4$ dimensionless parameters. These are called Pi products, since they are suitable products of the dimensional parameters. In the particular case of statement (1), suitable Pi products are:

$$\begin{aligned}
\Pi_1 &= \frac{L'}{\frac{1}{2}\rho_\infty V_\infty^2\, c} &&= c_\ell && \text{lift coefficient} \\
\Pi_2 &= \alpha &&= \alpha && \text{angle of attack} \\
\Pi_3 &= \frac{\rho_\infty V_\infty\, c}{\mu_\infty} &&= Re && \text{Reynolds number} \\
\Pi_4 &= \frac{V_\infty}{a_\infty} &&= M_\infty && \text{Mach number}
\end{aligned}$$

The dimensionless form of statement (1) then becomes

$$c_\ell \;=\; \bar{f}(\alpha, Re, M_\infty) \tag{2}$$

We see that the original 6 dimensional parameters which influence L' has been reduced to only 3 dimensionless parameters which influence c_ℓ.

Benefits of non-dimensionalization

The reduction of parameter count is potentially a huge simplification. Consider an exaustive lift-measurement experiment where the effect of all parameters is to be determined. Let's assume that in this experiment we need to give each parameter 5 distinct values in order to adequately ascertain its effect on the lift. If we work with the 6 dimensional parameters in statement (1), then the number of possible parameter combinations and experimental runs required is $5^6 = 15625$ (!). But if we work with the 3 dimensionless parameters in statement (2), the number of parameter combinations and experimental runs is only $5^3 = 125$, which is more than a hundredfold reduction in effort. Nondimensionalization is clearly a powerful technique for minimizing experimental effort.

The benefits of non-dimensionalization also extend to theoretical work. Deducing a statement such as (2) at the outset can be useful to guide subsequent detailed analysis. Theoretical results are also usually more concise and clear when presented in dimensionless form.

Derivation of dimensionless forms

Anderson 1.7 has details on how the Pi product combinations can be derived for any complex situation using linear algebra. In many cases, however, the products can be obtained by physical insight, or perhaps by inspection. Several rules can be applied here:

- Any parameter which is already dimensionless, such as α, is automatically one of the Pi products.
- If two parameters have the same units, such as V_∞ and a_∞, then their ratio (M_∞ in this case) will be one of the Pi products.
- A power or simple multiple of a Pi product is an acceptable alternative Pi product. For example, $(V_\infty/a_\infty)^2$ is an acceptable alternative to V_∞/a_∞, and $\rho_\infty V_\infty^2$ is an acceptable alternative to $\frac{1}{2}\rho_\infty V_\infty^2$. Which particular forms are used is a matter of convention.
- Combinations of Pi products can replace the originals. For example, the 3rd and 4th products in the example could have been defined as

$$\Pi_3 = \frac{\rho_\infty a_\infty c}{\mu_\infty} = Re/M_\infty$$

$$\Pi_4 = \frac{V_\infty}{a_\infty} = M_\infty$$

  which is workable alternative, but perhaps less practical, and certainly less traditional.

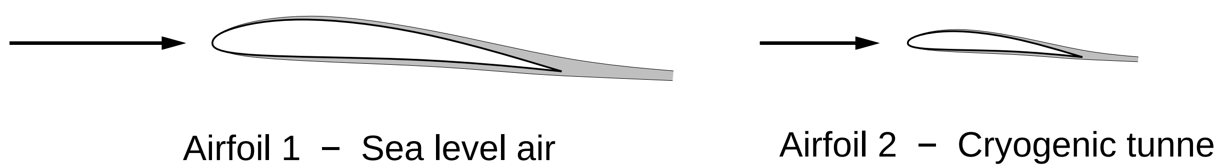
Dynamic Similarity

It is quite possible for two differently-sized physical situations, with different dimensional parameters, to nevertheless reduce to the same dimensionless description. The only requirement is that the corresponding Pi products have the same numerical values.

Airfoil flow example

Consider two airfoils which have the same shape and angle of attack, but have different sizes and are operating in two different fluids. Let's omit the $()_\infty$ subscript for clarity.

Airfoil 1 (sea level)	Airfoil 2 (cryogenic tunnel)
$\alpha_1 = 5°$	$\alpha_2 = 5°$
$V_1 = 210\text{m/s}$	$V_2 = 140\text{m/s}$
$\rho_1 = 1.2\text{kg/m}^3$	$\rho_2 = 3.0\text{kg/m}^3$
$\mu_1 = 1.8 \times 10^{-5}\text{kg/m-s}$	$\mu_2 = 1.5 \times 10^{-5}\text{kg/m-s}$
$a_1 = 300\text{m/s}$	$a_2 = 200\text{m/s}$
$c_1 = 1.0\text{m}$	$c_2 = 0.5\text{m}$



The Pi products evaluate to the following values.

Airfoil 1	Airfoil 2
$\alpha_1 = 5°$	$\alpha_2 = 5°$
$Re_1 = 1.4 \times 10^7$	$Re_2 = 1.4 \times 10^7$
$M_1 = 0.7$	$M_2 = 0.7$

Since these are also the arguments to the $\bar{f}$ function, we conclude that the c_ℓ values will be the same as well.

$$\begin{aligned} \bar{f}(\alpha_1, Re_1, M_1) &= \bar{f}(\alpha_2, Re_2, M_2) \\ c_{\ell_1} &= c_{\ell_2} \end{aligned}$$

When the nondimensionalized parameters are equal like this, the two situations are said to have *dynamic similarity* One can then conclude that any other dimensionless quantity must also match between the two situations. This is the basis of wind tunnel testing, where the flow about a model object duplicates and can be used to predict the flow about the full-size object. The prediction is correct only if the model and full-size objects have dynamic similarity.