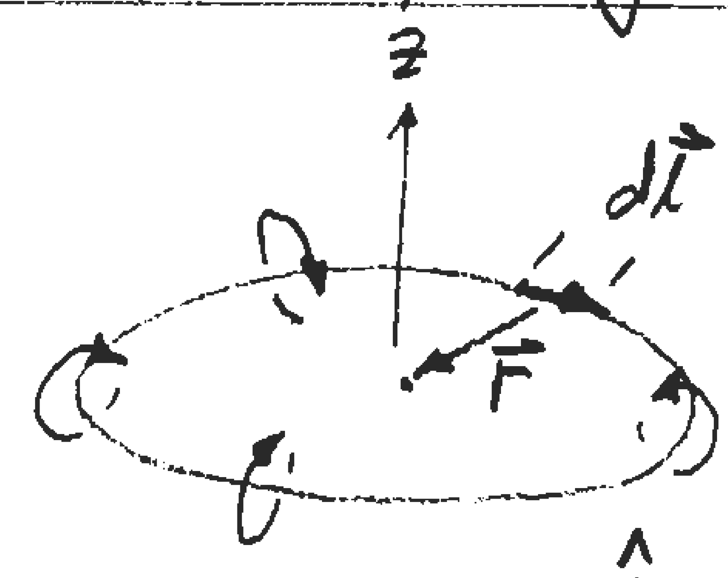
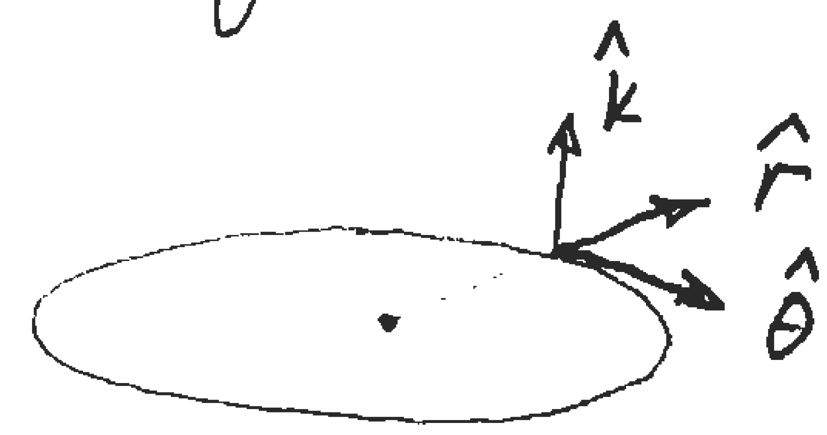
UE Fluids | F6 Solution | Spring '04

(Anderson p 416)

1. $\vec{V} = \dfrac{\Gamma}{4\pi}\displaystyle\int \dfrac{d\vec{l}\times\vec{r}}{r^3}$

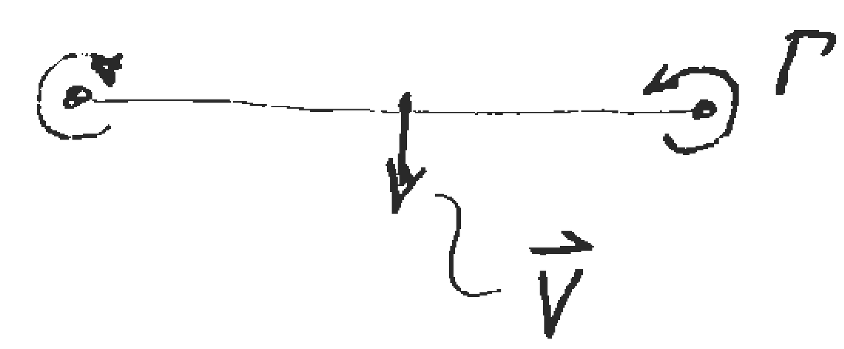


$d\vec{l} = R\,d\theta\,\hat{\theta}$, $\vec{r} = -R\hat{r}$, $r = R$



$d\vec{l}\times\vec{r} = -R^2 d\theta\;\hat{\theta}\times\hat{r} = -R^2 d\theta\;\hat{k}$

$$\boxed{\vec{V} = \frac{\Gamma}{4\pi}\int_0^{2\pi}\frac{-R^2 d\theta}{R^3}\,\hat{k} = -\frac{\Gamma}{2R}\,\hat{k}}$$



2. Now we have

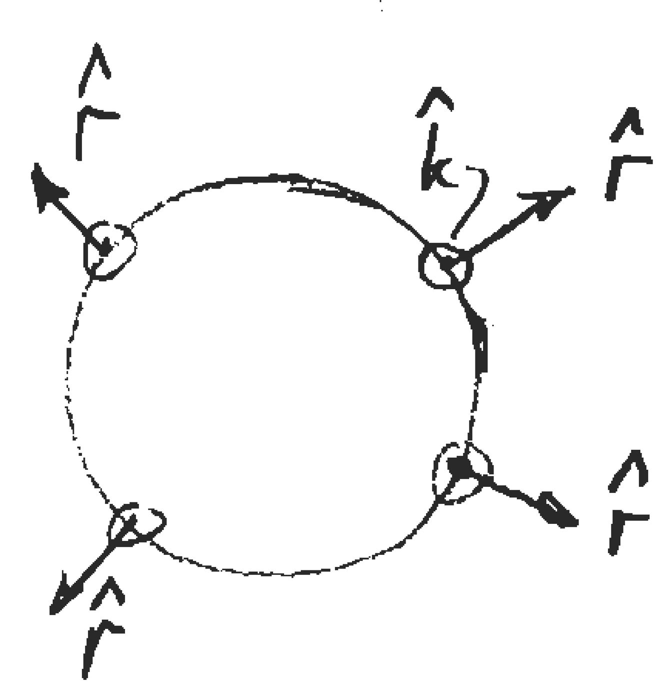
$d\vec{l} = R\,d\theta\,\hat{\theta}$, $\vec{r} = -R\hat{r} + A\hat{k}$

$r^3 = (R^2 + A^2)^{3/2}$

$$d\vec{l}\times\vec{r} = \left(-R^2(\hat{\theta}\times\hat{r}) + RA(\hat{\theta}\times\hat{k})\right)d\theta = \left(-R^2\hat{k} - RA\hat{r}\right)d\theta$$

Note that $\hat{k}$ is constant, but $\hat{r}$ depends on θ (varies around circle)

$$\vec{V} = \frac{\Gamma}{4\pi}\int_0^{2\pi}\frac{-R^2\hat{k} - RA\hat{r}}{(R^2+A^2)^{3/2}}\,d\theta$$



$$\vec{V} = -\frac{\Gamma}{2}\frac{R^2}{(R^2+A^2)^{3/2}}\hat{k} + \frac{\Gamma}{4\pi}\frac{-RA}{(R^2+A^2)^{3/2}}\int_0^{2\pi}\hat{r}\,d\theta$$

But we note that $\displaystyle\int_0^{2\pi}\hat{r}\,d\theta = 0$, since $\hat{r}$ cancels when integrated around perimeter.

$$\boxed{\vec{V} = -\frac{\Gamma}{2}\frac{R^2}{(R^2+A^2)^{3/2}}\hat{k}}$$